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Complexes discrets pour les fluides incompressibles.

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Le temps est l'image mobile de l'éternité immobile. -Platoon

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Chapitre 1

Présentation.

L'objectif de cette thèse est d'étudier l'application du calcul extérieur à la résolution numérique des équations de la mécanique des fluides. L'application du calcul extérieur aux problèmes numériques, notamment au travers des éléments finis du calcul extérieur est un sujet très actif et ouvre la voie à de nombreux développements dans le domaine des éléments finis (voir par exemple [50, 24, 67]). Bien que majoritairement employé pour des problèmes provenant de l'électromagnétisme (voir [5]), le formalisme est également adapté pour traiter bien d'autres domaines de la physique (voir [34]) dont les équations de Navier-Stokes et présente un réel intérêt pour l'élaboration de méthodes les résolvant. Le principal atout du calcul extérieur réside dans la préservation de la structure algébrique, géométrique et topologique (voir 1.2.2) du problème considéré. Nous allons donc commencer par motiver l'intérêt de cette propriété de conservation pour les équations des fluides. Nous donnerons ensuite une rapide introduction au formalisme du calcul extérieur puis à son application au domaine des fluides. Enfin nous présenterons les travaux réalisés au cours de cette thèse. Le chapitre 3 est accepté pour publication [64]. Les chapitres 2 et 4 sont soumis [68, 65].

1.1 Historique.

L'idée d'utiliser une discrétisation basée sur la structure topologique et algébrique des espaces continus n'est pas nouvelle. Il est difficile d'être exhaustif sur les origines de cette idée qui remonte au moins à Whitney et Courant, Cf. [50, chapitre 7.8]. On peut par exemple citer la méthode du complexe topologique [1] dès 1977, ou plus récemment les méthodes de calcul extérieur discret [14]. Historiquement le calcul extérieur a été utilisé en analyse numérique pour l'électromagnétisme [5, 50] puis pour l'élasticité [9, 16]. L'utilisation du calcul extérieur pour la mécanique des fluides est plus récente. Il existe des formulations compatibles [13, 33] mais la vitesse du fluide s'avère être beaucoup plus subtile à traiter notamment à cause de sa régularité et des conditions de bord [27]. Différentes approches on été explorées dans la littérature : soit en conservant le complexe de De Rham comme cela est développé au chapitre 3, soit en développant des complexes plus réguliers. Cette seconde approche apporte de nombreuses difficultés techniques. À notre connaissance, les premières discrétisations utilisant des éléments finis en dimension 2 datent de l'article [11] et on été complétées une décennie plus tard [31, 32]. En dimension 3 on peut retrouver des constructions réalisées à la même époque mais demandant des conditions particulières sur le maillage [15] ou n'ayant pas exactement la régularité $\boldsymbol{H}^1$ pour la vitesse [20]. La première construction utilisant des éléments finis conformes remonte à [41] et requiert l'utisation de polynômes de degrés élevés. Le sujet reste très actif avec d'autres constructions récentes utilisant des éléments finis (telle que [69]) ou de façon plus systématique avec le développement d'outils pour la discrétisation de complexes différentiels de haute régularité (Cf. [51, 58]) ou encore d'autres méthodes telles que les éléments virtuels [63] ou des méthodes hybrides comme celle développée au chapitre 4. Pour une perspective historique plus complète, nous renvoyons par exemple à [25] ou à [37].

1.2 La mécanique des fluides.

1.2.1 Les équations.

Les équations de Navier-Stokes traduisent simplement la conservation de la quantité de mouvement, du volume et de la masse. Nous nous intéressons à une classe particulière d'équations où nous supposons le fluide incompressible. Cela décrit très bien les liquides newtoniens tel que l'eau ou les gaz à faible vitesse et offre donc un énorme champ d'applications. Sous cette contrainte la dynamique ne dépendra plus que de la viscosité cinématique ν et des forces extérieures f; les équations se simplifient en

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + \nabla p + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= f, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

où les inconnues sont la vitesse $\boldsymbol{u}$ et la pression p.

Bien entendu, de très nombreuses méthodes on été développées pour la résolution numérique de ces équations. On peut notamment retrouver dans les méthodes les plus classiques des méthodes d'éléments finis. Un exemple standard est celui des éléments de Taylor-Hood (voir par exemple [3]) qui recherchent la vitesse et la pression respectivement comme des polynômes par morceau globalement continus de degré 2 et 1. On peut voir que la méthode fonctionne et donne un ordre d'approximation optimal. Il est donc naturel de se demander l'intérêt de développer d'autres méthodes. Cette méthode échoue cependant à préserver certaines propriétés, ce qui se traduit par l'apparition d'erreurs non négligeables. Un exemple est que la vitesse d'un fluide incompressible est insensible à l'ajout d'un gradient aux forces s'exerçant dessus. Cela peut se voir dans les équations où la pression peut « absorber » de telles forces. En particulier les forces dérivant d'un potentiel telle que la gravité ne peuvent mettre en mouvement un fluide, ce qui correspond bien à l'intuition. On peut considérer le cas d'un verre d'eau au repos, on ne s'attend à l'apparition d'aucun mouvement; cependant lorsque l'on réalise cette simulation avec des éléments de Taylor-Hood, de tels courants se créent. Ce type de problème est présenté en détail dans [48] et [56], celui du verre d'eau provenant de la seconde référence. Le problème ici est que l'espace discret, utilisant un gradient inexact, ne parvient pas à préserver cette propriété de l'espace continu. Une méthode respectant cette propriété est dite robuste en pression, ce qui est une qualité très avantageuse (voir [48, 56]).

1.2.2 Préservation de la structure.

L'espace discret ne peut pas préserver toutes les propriétés du continu. Cependant toutes les propriétés ne sont pas nécessairement d'intérêt. Il faut ainsi choisir les propriétés les plus importantes et utiliser des structures et formulations discrètes les préservant.

En mécanique des fluides, les propriétés nous intéressant seront principalement l'énergie et la contrainte sur la divergence. Nous verrons également d'autres formes de structures préservées avec, par exemple, la reproduction des états dégénérés présents pour certaines géométries (plus précisément du noyau du laplacien, voir les chapitres 1.4.2 et 2.8) .

La préservation de ces structures présente, non seulement un aspect algébrique évident en demandant l'annulation exacte de certaines quantités dans les formulations, mais également un aspect très géométrique. La contrainte de divergence nulle demande que toute entrée de matière dans un volume Ω soit compensée par une sortie. Cela peut se formaliser avec le théorème de la divergence : Si $\boldsymbol{u}$ est la vitesse du fluide et $\boldsymbol{n}$ le vecteur sortant normal à la surface $\partial\Omega$ de Ω alors

$$\int_\Omega \operatorname{div} \boldsymbol{u} = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \,. \tag{1.1}$$

Ainsi au niveau discret il faut que sur chaque cellule, la somme des flux entrants et sortants s'annule. Cela suffirait à imposer la contrainte si nous ne prenions que des fonctions constantes par cellule. Pour des espaces plus riches il faut imposer plus de conditions, mais l'idée reste valide et, raisonner en termes de flux normaux semble être prometteur. Cependant la direction de ces flux dépend de chaque cellule et, à moins de travailler uniquement avec des cellules cubiques, elle n'aura pas grand chose à voir avec le choix de coordonnées cartésiennes. On va donc rechercher une formulation plus intrinsèque du problème.

Il existe également d'autres formules d'intégration par partie reliant les variations à l'intérieur d'une cellule avec le comportement sur son bord. Ainsi le rotationnel de la vitesse va dépendre de la composante tangentielle de la vitesse au bord, au travers de la formule

$$\int_{\Omega} \mathbf{curl}\, \boldsymbol{u} = -\int_{\partial\Omega} \boldsymbol{u} \times \boldsymbol{n}\,. \tag{1.2}$$

Le théorème de la divergence qui est à la base de l'interprétation géométrique admet un formalisme extrêmement bien adapté : celui du calcul extérieur. Ici toutes les formules d'intégration par parties vectorielles telles que (1.1) ou (1.2) prennent une unique et même forme, la formule de Stokes :

$$\int_{\Omega} \mathrm{d}\,\omega = \int_{\partial\Omega} i^{*}\omega\,. \tag{1.3}$$

Bien que l'expression soit simple, elle fait appel à des objets avancés qui sont détaillés dans la section 1.3.2.

1.3 Une brève introduction au calcul extérieur.

L'introduction donnée ici au calcul extérieur est sommaire, une introduction plus détaillée peut être trouvée par exemple dans [39] ou dans [50] pour le cadre spécifique des éléments finis.

1.3.1 L'algèbre extérieure.

Le calcul extérieur est une théorie de géométrie différentielle. Il va donc s'intéresser aux déformations d'objets sur un domaine.

Soit une variété différentielle Ω de dimension n. En calcul vectoriel les objets considérés en chaque point $\boldsymbol{x}$ de Ω sont les scalaires (éléments de $\mathbb{R}$) et les vecteurs (éléments de $\mathbb{R}^n$). L'algèbre extérieure est construite à partir de ces objets et les généralise. Soit $\{\frac{\partial}{\partial x^i}\}_{1\leq i\leq n}$ une base de $T_{\boldsymbol{x}}\Omega$ l'espace tangent au point $\boldsymbol{x}$, nous définissons l'espace des covecteurs (ou 1-formes) comme l'espace dual de celui des vecteurs; ainsi une base est donnée par l'ensemble des formes linéaires $\{dx^i\}_{1\leq i\leq n}$ définies par

$$dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j\,.$$

L'idée principale est alors de ne pas se limiter aux formes linéaires mais de considérer l'ensemble des formes multilinéaires alternées. C'est à dire des formes multilinéaires s'annulant lorsqu'elles sont appliquées à des ensembles de vecteurs linéairement dépendants. Nous notons l'ensemble des formes k-linéaires alternées Alt^k.

Une première observation importante est que la condition d'alternation limite grandement la dimension des formes multilinéaires. Il apparait déjà qu'il ne peut exister d'ensemble libre de plus de n éléments, ainsi Alt^k est trivial pour $k > n$. Avec un peu plus de manipulations, on peut observer que :

$$\dim \mathrm{Alt}^k = \binom{n}{k} .$$

L'algèbre extérieure Alt est alors simplement la collection de ces formes graduées par leur degré :

$$\mathrm{Alt} := \bigoplus_{k=0}^{n} \mathrm{Alt}^k .$$

Cette algèbre est munie d'un produit associatif compatible avec le degré, appelé produit extérieur et noté $\wedge$. Pour $\alpha \in \mathrm{Alt}^i$ et $\beta \in \mathrm{Alt}^j$, il renvoie une forme $i + j$-linéaire alternée $\alpha \wedge \beta \in \mathrm{Alt}^{i+j}$ définie par

$$(\alpha \wedge \beta)(v_1, \dots, v_{i+j}) := \sum_{\sigma} \mathrm{sign}(\sigma)\alpha(v_{\sigma_1}, \dots, v_{\sigma_i})\beta(v_{\sigma_{i+1}}, \dots, v_{\sigma_{i+j}})$$

où σ parcourt le sous ensemble des permutations à $i + j$ éléments $\mathfrak{S}_{i+j}$ vérifiant $\sigma_1 < \cdots < \sigma_i$, $\sigma_{i+1} < \cdots < \sigma_{i+j}$. Il s'agit simplement du produit tensoriel des deux applications rendu alterné. On peut montrer qu'il suffit à engendrer l'algèbre extérieure à partir des 1-formes dans le sens où $\{\alpha_1 \wedge \cdots \wedge \alpha_k \ : \ \alpha_i \in \mathrm{Alt}^1\}$ est générateur de Alt^k.

L'espace des k-formes peut également être vu comme le dual d'un autre espace. Une forme k-linéaire alternée sur les vecteurs est ainsi une forme linéaire sur un autre objet, appelé un k-vecteur. Le produit extérieur se transporte alors par dualité sur les k-vecteurs, de sorte que $\{\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\}_{i<j}$ soit la base duale de $\{dx^i \wedge dx^j\}_{i<j}$.

Le choix de se restreindre à des formes alternées a une interprétation géométrique plus apparente en regardant les k-vecteurs. En effet, supposons que l'espace des vecteurs soit muni d'un produit scalaire, un k-vecteur pur s'écrit sous la forme $v_1 \wedge \cdots \wedge v_k$ et s'identifie à un ensemble de k vecteurs. On considère alors le k-parallélotope dont les côtés sont donnés par ces k vecteurs. Le k-vecteur peut alors être interprété géométriquement comme une « direction » (un sous-espace de dimension k où vit le parallélotope) et le volume orienté du parallélotope. Deux k-vecteurs purs sont les mêmes si et seulement si leurs parallélotopes vivent dans le même espace et ont même volume. Cette construction est illustrée dans la figure 1.1. La structure alternée des k-vecteurs correspond ici au simple fait que le

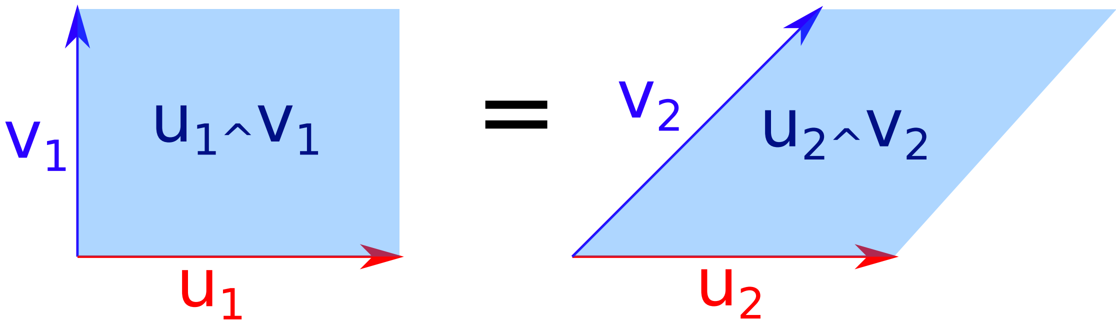


Figure 1.1: Deux paires de vecteurs ayant un même produit extérieur.

volume d'un parallélotope avec deux cotés colinéaires est nul.

Un cas particulier apparaît pour les n-vecteurs : il ne peut exister qu'un seul n-vecteur à multiples scalaires près. Cela veut également dire que tous les n-vecteurs sont purs. Géométriquement la direction est forcément l'espace entier et un n-vecteur est alors uniquement défini par son volume.

Supposons à présent que Ω soit une variété Riemannienne. En tout point $\boldsymbol{x} \in \Omega$ l'espace tangent $T_{\boldsymbol{x}}\Omega$ est alors muni d'un produit scalaire $\langle \cdot, \cdot \rangle$ de matrice g dans la base $\{\frac{\partial}{\partial x^i}\}_{1 \leq i \leq n}$. L'espace dual des 1-formes est alors naturellement muni d'un produit scalaire $\langle \cdot, \cdot \rangle$ donné par la matrice g^{-1} dans la base duale $\{dx^i\}_{1 \leq i \leq n}$. On peut alors étendre ce produit scalaire sur les k-formes à l'aide du déterminant de Gram. Pour $u = u^1 \wedge \cdots \wedge u^k, v = v^1 \wedge \cdots \wedge v^k \in \mathrm{Alt}^k$ des k-formes pures, on définit :

$$\langle u, v \rangle := \det \left[(\langle u^i, v^j \rangle)_{1 \leq i,j \leq k} \right] .$$

La définition s'étend par linéarité sur l'espace des k-formes. Nous avons ainsi un produit scalaire sur l'espace Alt^k.

Comme vu précédement, l'espace des n-formes Alt^n est de dimension 1; ainsi en fixant une n-forme particulière vol_n appelée forme de volume, nous pouvons identifier Alt^n avec $\mathbb{R}$ puisque toute n-forme s'écrit comme un multiple de vol_n. Le produit scalaire sur Alt^n permet presque de sélectionner sans choix une forme de volume en prenant une n-forme de norme 1. Il n'y a alors plus que deux possibilités :

$$\mathrm{vol}_n := \pm \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n .$$

Le choix du signe revient au choix de l'orientation de l'espace.

Exemple:

Sur un espace Euclidien tel que $\mathbb{R}^2$ la métrique est triviale et si (dx, dy) est une base directe alors la forme de volume est simplement :

$$\mathrm{vol}_{\mathbb{R}^2} = dx \wedge dy .$$

Si l'on munit ce même espace de coordonnées polaires (r, θ) alors la métrique devient

$$g_P = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} ,$$

et la forme de volume est :

$$\mathrm{vol}_P = \sqrt{r^2} dr \wedge d\theta = r dr \wedge d\theta \,.$$

Grâce à l'identification $\mathrm{Alt}^n \approx \mathbb{R}$, nous avons deux opérations linéaires différentes agissant sur une k-forme et renvoyant un scalaire: le produit scalaire avec une autre k-forme et le produit extérieur avec une $n-k$-forme. A la manière de l'identification de Riesz, cela permet de construire une identification entre les k-formes et les $n-k$-formes : l'étoile de Hodge noté $\star$. Formellement pour tout $\beta \in \mathrm{Alt}^k$, $\star\beta \in \mathrm{Alt}^{n-k}$ est tel que :

$$\forall \alpha \in \mathrm{Alt}^k, \quad \alpha \wedge (\star\beta) := \langle \alpha, \beta \rangle \mathrm{vol}_n \,. \tag{1.4}$$

Exemple:
Explicitons quelques calculs sur $\mathbb{R}^3$ muni d'une base directe (dx, dy, dz). De manière générale on a

$$\star 1 = \mathrm{vol}_{\mathbb{R}^3} = dx \wedge dy \wedge dz \,,$$

$$\star(dx \wedge dy \wedge dz) = 1 \,.$$

Il ne reste plus qu'à le calculer sur les 1-formes et 2-formes :

$$\begin{aligned} \star dx &= dy \wedge dz, & \star(dy \wedge dz) &= dx \,, \\ \star dy &= -\, dx \wedge dz, & \star(dx \wedge dz) &= -\, dy \,, \\ \star dz &= dx \wedge dy, & \star(dx \wedge dy) &= dz \,. \end{aligned}$$

1.3.2 Géométrie locale.

Après avoir défini la structure ponctuelle de l'algèbre extérieure nous pouvons regarder la structure locale. Les objets traités par le calcul extérieur sont des champs de k-formes, c'est à dire une association en tout point $\boldsymbol{x} \in \Omega$ d'une k-forme dont l'expression en coordonnées possède une certaine régularité (par exemple continue ou lisse). L'ensemble des champs de k-formes (appelé simplement k-formes dans la suite) est noté $\Lambda^k(\Omega)$. La régularité sera précisée avec la notation, par exemple $L^2\Lambda^k(\Omega)$, si l'on veut spécifier que la forme est de carré intégrable.

La plupart des opérations se définissent ponctuellement, cependant il y a également 3 opérations importantes apparaissant dans la formule de Stokes (1.3) dépendant du comportement local des champs. Tout d'abord, l'image réciproque ou tirée en arrière [1] par une application ϕ noté ϕ^* permet de transporter les objets d'un espace à un autre. Soit ϕ une application de Ω vers Ω', sa différentielle $D\phi$ est alors une application de $T\Omega$ vers $T\Omega'$. L'image réciproque d'un champ de k-forme ω sur Ω' est le champ $\phi^*\omega$ sur Ω tel que

$$\forall \boldsymbol{x} \in \Omega, \forall v_1, \ldots, v_k \in T_{\boldsymbol{x}}\Omega, \quad \phi^*\omega_{\boldsymbol{x}}(v_1, \ldots, v_k) := \omega_{\phi(\boldsymbol{x})}(D\phi\, v_1, \ldots, D\phi\, v_k)\,.$$

Exemple:
L'application $\phi : (r, \theta) \to (x, y) = (r\cos\theta, r\sin\theta)$ transforme les coordonnées polaires en coordonnées cartésiennes. Son image réciproque va donc transporter les structures données en coordonnées cartésiennes vers les coordonnées polaires. Explicitement :

$$\begin{aligned}\phi^* dx(v_r \frac{\partial}{\partial r} + v_\theta \frac{\partial}{\partial \theta}) &= dx(v_r \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + v_\theta \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + v_r \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + v_\theta \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}) \\ &= v_r \frac{\partial x}{\partial r} + v_\theta \frac{\partial x}{\partial \theta} \\ &= v_r \cos\theta - v_\theta\, r\sin\theta.\end{aligned}$$

Ainsi :

$$\phi^* dx = \cos\theta\, dr - r\sin\theta\, d\theta, \text{ de même } \phi^* dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

On peut alors calculer l'expression de la forme de volume

$$\begin{aligned}\mathrm{vol}_P = \phi^* dx \wedge dy &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr \\ &= r(\cos^2\theta + \sin^2\theta)\, dr \wedge d\theta \\ &= r\, dr \wedge d\theta\,.\end{aligned}$$

On retrouve bien sûr l'expression de l'élément de surface en coordonnées polaires.

Ensuite nous donnons un sens à la notion d'intégration. Dans le cas le plus simple de l'intégrale sur un ouvert U de $\mathbb{R}^n$ d'une n-forme $\omega = f(x)\mathrm{vol}_n$, on définit :

$$\int_U \omega := \int_U f(x)\, dx^1 \ldots dx^n\,.$$

[1] « Pullback » en anglais.

Dans le cas plus général de l'intégrale d'une k-forme sur une k-variété orientable, l'existence d'un atlas assure l'existence de fonction ϕ_i de $\mathbb{R}^k$ vers un recouvrement d'ouvert U_i orienté positivement. On peut alors multiplier la k-forme par une partition de l'unité pour obtenir une somme de k-formes chacune supportée sur un seul ouvert U_i, et considérer l'image réciproque par ϕ_i pour revenir au cas d'intégrale sur un espace euclidien. On peut alors vérifier que cette définition est bien indépendante du choix de l'atlas (voir [12]).

Enfin nous définissons l'opérateur fondamental du calcul extérieur : la dérivée extérieure notée d. C'est un opérateur linéaire de degré 1, c'est à dire transformant les k-formes en $(k+1)$-formes. Il est défini sur les 0-formes (les champs scalaires) comme la différentielle, explicitement pour $f \in \Lambda^0(\Omega)$,

$$\mathrm{d}\, f := \sum_{i=1}^{k} \frac{\partial f}{\partial x^i}\, dx^i \,.$$

Sa définition s'étend à toute k-forme à l'aide de deux propriétés le caractérisant:

- $\mathrm{d}(\mathrm{d}\,\alpha) = 0$, pour toute k-forme α.
- $\forall \alpha \in \Lambda^k(\Omega), \forall \beta \in \Lambda^p(\Omega),\ \mathrm{d}(\alpha \wedge \beta) := \mathrm{d}\,\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\,\beta.$

On peut remarquer que l'image réciproque commute avec la différentielle extérieure.

Nous utiliserons également l'adjoint δ de la différentielle d appelé codifférentielle. Cet opérateur peut s'exprimer à l'aide de l'étoile de Hodge ainsi :

$$\delta \,:\, \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega), \quad \delta := (-1)^{n(k-1)+1} \star \mathrm{d} \star . \tag{1.5}$$

La formule de Stokes (1.3) peut alors être énoncée: soit Ω une n-variété différentielle lisse à bord et ω une $(n-1)$-forme. Alors pour i l'inclusion $\partial\Omega \hookrightarrow \Omega$ on a:

$$\int_\Omega \mathrm{d}\,\omega = \int_{\partial\Omega} i^* \omega \,. \tag{1.6}$$

1.3.3 Vecteur « mandataire ».

Les vecteurs « mandataires[2] » font le lien entre la dérivée extérieure et les opérateurs usuels : grad, curl et div de l'analyse vectorielle. Lorsque Ω est un espace euclidien, les 1-formes s'identifient naturellement aux vecteurs à l'aide de l'identification de Riesz; de plus grâce à l'étoile de Hodge les $(n-1)$-formes s'identifient aux 1-formes, donc aux vecteurs. De même les 0-formes sont les scalaires et les n-formes s'identifient aux 0-formes. Ainsi nous avons 4 espaces du calcul extérieur

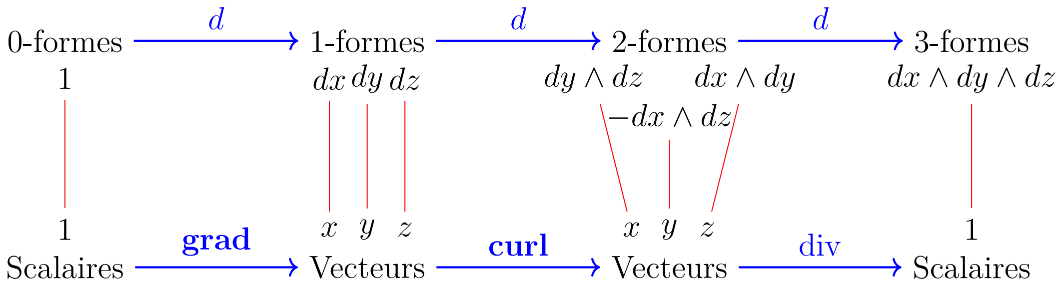


Figure 1.2: Identification entre les vecteurs et les formes en dimension 3.

naturellement identifiés avec ceux du calcul vectoriel, à savoir les 0, 1, $n-1$ et n formes.

Lorsque Ω est de dimension 3, il n'y a que 4 espaces dans l'algèbre extérieure, ainsi ils peuvent tous être identifiés avec le calcul vectoriel comme présentés dans la figure 1.2. En dimension 2 il n'y a que 3 espaces, les espaces $\Lambda^1(\Omega)$ et $\Lambda^{n-1}(\Omega)$ étant les mêmes. Cela va donc donner deux identifications différentes en termes de calcul vectoriel présentées dans la figure 1.3. Un fait remarquable est que ces identifications transforment la dérivée extérieure en opérateurs différentiels usuels de sorte que les diagrammes des figures 1.3 et 1.2 commutent.

Exemple:
Soit une 1-forme $adx + bdy + cdz$ en dimension 3. Suivant l'identification de la figure 1.2, cette forme est représentée par le vecteur (a, b, c). Appliquons à présent la différentielle extérieure :

$$\begin{aligned}
\mathrm{d}(adx + bdy + cdz) = & \cancel{\frac{\partial a}{\partial x} dx} \wedge dx + \frac{\partial a}{\partial y} dy \wedge dx + \frac{\partial a}{\partial z} dz \wedge dx \\
& + \frac{\partial b}{\partial x} dx \wedge dy + \cancel{\frac{\partial b}{\partial y} dy} \wedge dy + \frac{\partial b}{\partial z} dz \wedge dy \\
& + \frac{\partial c}{\partial x} dx \wedge dz + \frac{\partial c}{\partial y} dy \wedge dz + \cancel{\frac{\partial c}{\partial z} dz \wedge} dz \\
= & \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} \right) dy \wedge dz - \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x} \right) dx \wedge dz + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy
\end{aligned}$$

Toujours suivant la même identification, la 2-forme $\left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right) dy \wedge dz - \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right) dx \wedge dz + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$ est représentée par le vecteur $(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}, \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}, \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})$. Le représentant de $\mathrm{d}(adx + bdy + cdz)$ est

[2] « Proxy vector » en anglais.

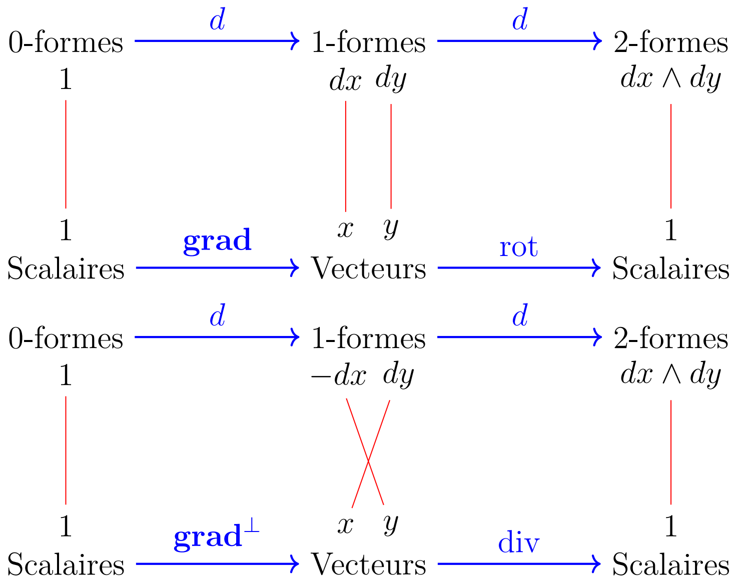


Figure 1.3: Deux identifications possibles entre les vecteurs et les formes en dimension 2.

donc bien $\mathbf{curl}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

L'équivalence entre les deux formalismes n'est vraie que pour les cas simples (bien qu'assez généraux) de la dimension inférieure à 3 avec une métrique triviale. Au-delà le calcul extérieur offre une véritable généralisation avec l'apparition d'espaces sans correspondance directe en dimension supérieure à 4 et un traitement plus naturel des espaces courbes. En effet lorsque la métrique n'est pas triviale, elle va intervenir lors de l'identification entre les vecteurs et les 1-formes et les formules pour passer de l'un à l'autre sont plus compliquées. La formulation à l'aide du calcul extérieur va alors offrir un très grand avantage puisque les formules différentielles sont déjà valides pour des métriques arbitraires. Cela dit, dans ce travail nous restons dans le cadre des espaces euclidiens [3] $\mathbb{R}^2$ et $\mathbb{R}^3$.

[3] La généralisation de la méthode aux variétés courbes est une perspective fascinante.

1.4 Mécanique des fluides et calcul extérieur.

1.4.1 Reformulation des équations.

Nous regardons à présent plus en détail l'application du calcul extérieur à la mécanique des fluides incompressibles. Pour commencer nous allons réécrire les équations de Navier-Stokes à l'aide des opérateurs différentiels obtenus dans l'équivalence précédente. La formulation de Lamb $(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\mathbf{curl}\,\boldsymbol{u})$ et l'écriture du laplacien en rotationnel $\Delta\boldsymbol{u} = \mathbf{grad}\operatorname{div}\boldsymbol{u} - \mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}) = -\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u})$ dans le cas incompressible permettent de séparer les parties provenant de pressions internes de celles engendrées par les rotations :

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} + \nu\,\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}) + \mathbf{grad}\,P + (\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u} &= f,\\ \operatorname{div}\boldsymbol{u} &= 0,\end{aligned}$$

avec $P := \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + p$ la pression de Bernoulli. On peut déjà voir apparaître plus clairement la distinction géométrique entre les différents éléments de la dynamique. En particulier, toutes les pertes d'énergie dues à la viscosité se décrivent à l'aide du rotationnel.

Il y a deux possibilités pour traduire cette équation à l'aide des différentielles et codifférentielles extérieures. On peut soit voir la vélocité comme une 1-forme soit comme une 2-forme. Si $\boldsymbol{u}$ est une 1-forme alors $\mathbf{curl}\,\boldsymbol{u}$ est le représentant de $\mathrm{d}\,\boldsymbol{u}$, P doit être une 0-forme avec $\mathbf{grad}\,P$ le représentant de $\mathrm{d}\,P$ et les équations s'écrivent sous la forme :

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} + \nu\delta\,\mathrm{d}\,\boldsymbol{u} + \mathrm{d}\,P + \star\big((\star\,\mathrm{d}\,\boldsymbol{u})\wedge\boldsymbol{u}\big) &= f,\\ \delta\boldsymbol{u} &= 0.\end{aligned}$$

Et si $\boldsymbol{u}$ est une 2-forme alors $\operatorname{div}\boldsymbol{u}$ représente $\mathrm{d}\,\boldsymbol{u}$, P doit être une 3-forme et les équations s'écrivent sous la forme :

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} + \nu\,\mathrm{d}\,\delta\boldsymbol{u} + \delta P + \delta\boldsymbol{u}\wedge\star\boldsymbol{u} &= f,\\ \mathrm{d}\,\boldsymbol{u} &= 0.\end{aligned}$$

Ce choix a beaucoup d'importance pour l'interprétation mais surtout pour la discrétisation. Il sera discuté plus en détail dans la section 1.4.3.

1.4.2 Complexes différentiels.

Regardons à présent la discrétisation du calcul extérieur. Comme discuté dans la section 1.2.2 l'objectif principal de ce travail est de réaliser des discrétisations préservant la structure des équations. Les équations s'écrivent à partir de l'opérateur différentiel d et sa caractérisation algébrique principale est sa nilpotence : $\mathrm{d} \circ \mathrm{d} = 0$. Cette caractérisation combinée à sa compatibilité avec la graduation de l'espace représente exactement la structure d'un complexe. Un complexe est une suite d'espaces reliés par une suite d'opérateurs tel que l'image du précédent soit contenu dans le noyau du suivant. Avec la différentielle extérieure nous avons donc le complexe de De Rham :

$$\{0\} \xrightarrow{0} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathrm{curl}, \Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}.$$

Il y a de nombreux avantages à utiliser un complexe pour traiter des équations : il va offrir une norme naturelle avec la norme de graphe ($\|\boldsymbol{u}\|_{\boldsymbol{H}} := \|\boldsymbol{u}\|_{L^2} + \|\,\mathrm{d}\,\boldsymbol{u}\|_{L^2}$) qui restera simple à contrôler car sera limitée au premier ordre (puisque composer deux fois l'opérateur différentiel donne toujours 0). De plus cela permet facilement de coupler les systèmes avec d'autres s'exprimant dans le complexe. Un exemple important est celui des équations de Maxwell qui s'inscrivent également dans le complexe de De Rham et se couplent avec celles de Navier-Stokes dans l'étude de la magnétohydrodynamique. Enfin on peut se demander à quel point l'image d'un opérateur diffère du noyau du suivant (ou quelle est la cohomologie du complexe). Dans le cas du complexe de De Rham, cette question est bien comprise : le quotient est de dimension finie donnée par les nombres de Betti du domaine (nombre de composantes connexes, nombre de tunnels, nombre de bulles). Ces nombres dépendent uniquement de la topologie et sont très simples à calculer. Cela facilite l'extraction du noyau, ce qui permet d'obtenir des systèmes linéaires stables. Les éléments étant dans le noyau d'un opérateur sans être image du précédent [4] jouent un rôle important dans l'analyse et sont appelés *formes harmoniques*. On peut remarquer que cela est équivalent à être simultanément dans le noyau de la différentielle et de la codifférentielle, ce qui dans le complexe de De Rham, revient à être dans le noyau du laplacien, d'où le nom forme harmonique.

On va donc s'appuyer sur les complexes pour élaborer les espaces discrets. Cela va donner la notion de sous-complexes et est la fondation des éléments finis de calcul extérieur. Nous allons simplement évoquer les principaux concepts ici, de plus amples détails peuvent être trouvés dans [50]. Un sous-complexe doit être lui-même un complexe (donc la donnée d'espaces et d'opérateurs les reliant) possédant de bonnes propriétés d'approximation pour des raisons évidentes de convergence. En pratique on va naturellement choisir de prendre pour opérateur discret d_h la restriction de l'opérateur différentiel continu d à l'espace discret. Enfin la base d'un complexe différentiel étant la dérivée extérieure, la notion de morphisme entre deux complexes va être celle d'opérateur linéaires commutant avec la différentielle.

[4] Plus précisément il s'agit du quotient du noyau par l'image.

Le sous-complexe discret va donc devoir être connecté au complexe continu par des projecteurs π_h commutant avec la différentielle, voir [50, chapitre 5.2]. Ces quelques propriétés ont déjà des conséquences assez fortes sur le complexe. On sait par exemple que la cohomologie du complexe discret sera isomorphe à celle du continu et on peut majorer l'écart entre l'espace des formes harmoniques discrètes $\mathfrak{H}_h$ et continues $\mathfrak{H}$:

$$\max\left(\sup_{\phi\in\mathfrak{H},\|\phi\|=1}\inf_{\psi\in\mathfrak{H}_h}\|\phi-\psi\|,\ \sup_{\phi\in\mathfrak{H}_h,\|\phi\|=1}\inf_{\psi\in\mathfrak{H}}\|\phi-\psi\|\right)\leq\sup_{q\in\mathfrak{H},\|q\|=1}\|(\mathrm{Id}-\pi_h)\,q\|\,.$$

Du point de vue du calcul vectoriel, on cherche donc des espaces discrétisant correctement le gradient, le rotationel ou la divergence. De tels espaces ont déjà été développés (principalement pour la discrétisation de problème d'électromagnétisme). On peut ainsi citer les éléments de Raviart-Thomas, de Nédélec, ou de Brezzi-Douglas-Marini. Il est possible de les agencer correctement pour obtenir des complexes de tout degré polynomial. Cependant le calcul vectoriel passe à côté d'un fait remarquable: tous ces espaces peuvent être obtenus comme cas particuliers d'une seule et même famille lorsqu'ils sont exprimés dans le formalisme du calcul extérieur. Cette construction a été développée par Arnold [50, Chapter 7] et a conduit à la création d'un tableau les référençant [36] (à la manière du tableau périodique de Mendeleïev). L'origine de cette uniformité peut se voir d'une autre manière: tout comme la métrique de l'espace est déjà encodée dans les opérateurs du calcul extérieur, le type de continuité requise aux interfaces (tangentielle pour le rotationnel, normale pour la divergence) est lui aussi encodé dans les vecteurs mandataires et l'étoile de Hodge. Ainsi du point de vue du calcul extérieur, on demande simplement à ce que la trace soit continue indifféremment du degré de la forme.

Il existe tout de même une différence très importante entre le complexe continu et discret au niveau de la codifférentielle. Puisqu'elle est définie à l'aide d'adjoint sur un produit L^2, la codifférentielle va dépendre de l'espace où elle est définie et va donc grandement différer entre le cas continu et le cas discret. Le complexe de De Rham continu s'écrit :

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\mathrm{d}^1=\mathbf{grad}} & H(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{d}^2=\mathbf{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{d}^3=\mathrm{div}} & L^2(\Omega) \\ & & & & & & . \\ L^2_0(\Omega) & \xleftarrow[\delta^1=-\,\mathrm{div}]{} & H_0(\mathrm{div},\Omega) & \xleftarrow[\delta^2=\mathbf{curl}]{} & H_0(\mathrm{curl},\Omega) & \xleftarrow[\delta^3=-\,\mathbf{grad}]{} & H^1_0(\Omega) \end{array}$$

Il possède une symétrie exceptionnelle puisque, sur l'intersection de leurs domaines de définition respectifs, $\mathrm{d}^1=-\delta^3$, $\mathrm{d}^2=\delta^2$ et $\mathrm{d}^3=-\delta^1$. Cette propriété est très largement utilisée pour traduire les équations différentielles vectorielles dans le formalisme du calcul extérieur mais elle n'est plus respectée sur le complexe discret ($\delta_h^1 \neq -\operatorname{div}_h$, etc.).

1.4.3 Caractère dual des fluides.

On arrive ainsi à une question fondamentale qui est le choix de la forme pour la vitesse $\boldsymbol{u}$ du fluide. Puisque la codifférentielle discrète δ_h diffère de la codifférentielle continue δ, une erreur sera nécessairement commise lors de la discrétisation. Si l'on choisit de prendre la vitesse discrète $\boldsymbol{u}_h$ comme une 1-forme, on aura une vorticité discrète $\boldsymbol{\omega}_h = \mathrm{d}_h\, \boldsymbol{u}_h$ exacte puisque $\mathrm{d}_h\, \boldsymbol{u}_h = \mathrm{d}\, \boldsymbol{u}_h$ mais nous commettrons une erreur sur la divergence de $\boldsymbol{u}_h$ puisque $\delta_h \boldsymbol{u}_h \neq \delta \boldsymbol{u}_h$. Au contraire, en prenant $\boldsymbol{u}_h$ une 2-forme, on obtient une divergence exacte mais l'on introduit une erreur sur $\boldsymbol{\omega}_h$. Cela revient également à choisir de privilégier la composante normale (resp. tangentielle) de la trace de la vitesse vue comme une 2-forme (resp. 1-forme) aux interfaces.

Il faut donc choisir entre une incompressibilité exacte et la vorticité exacte qui va intervenir dans l'advection dans la formulation de Lamb. La formulation du laplacien en rotationnel ou bien la vorticité intervenant dedans étant impactée, on ne pourra malheureusement pas, dans les deux cas, conserver exactement le terme dissipatif. Il semble que $\boldsymbol{u}$ soit réellement à la fois une 1-forme et une 2-forme [2]. Cependant il ne semble pas possible de préserver simultanément le complexe et son adjoint au niveau discret. Dans ces travaux, j'ai dû choisir et ai très largement privilégié de considérer $\boldsymbol{u}$ comme une 2-forme. La condition d'incompressibilité exacte permettant notamment de dériver des propriétés importantes comme la robustesse en pression découplant l'erreur due à la vitesse de celle sur la pression, robustesse évoquée dans le chapitre 1.2.1.

1.5 Exposé général des résultats.

Dans le chapitre 2 nous avons commencé par un problème plus simple: le problème de Biot-Savart. Le problème consiste à réaliser un champ $\boldsymbol{u}$ avec un rotationnel $\boldsymbol{f}$, une divergence g prescrite et des conditions limites compatibles. Contrairement au problème de Stokes, le problème de Biot-Savart n'utilise que des opérateurs différentiels du premier ordre. On ne peut donc pas symétriser la formulation variationnelle et cela conduit à utiliser une formulation « asymétrique » telle que pour tous $\boldsymbol{v}$ et q dans des espaces convenables,

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{u}) \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,,$$
$$\int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, q = -\int_\Omega g\, q\,.$$

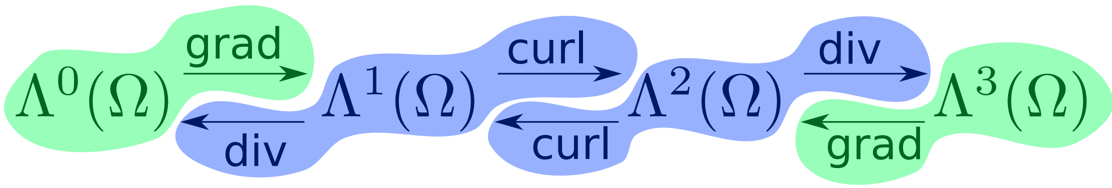


Figure 1.4: Répartition des problèmes de Hodge-Dirac sur le complexe de De Rham.

Du point de vue du calcul vectoriel, pour rendre le problème bien posé dans une formulation variationnelle, il faudrait déterminer les noyaux non triviaux de **curl** et div (l'adjoint de **grad**). Cependant si l'on utilise le formalisme du calcul extérieur, on se rend compte que ce problème apparaît naturellement deux fois dans le complexe de De Rham. On voit également apparaître deux fois un problème similaire demandant de déterminer un champ p avec un gradient $\boldsymbol{r}$ prescrit. Ces quatres problèmes s'avèrent être complémentaires, comme l'illustre la figure 1.4, et les poser simultanément restaure la symétrie de la formulation variationnelle : Trouver $p_1 \in \Lambda^0(\Omega)$, $\boldsymbol{u}_1 \in \Lambda^1(\Omega)$, $\boldsymbol{u}_2 \in \Lambda^2(\Omega)$, $p_2 \in \Lambda^3(\Omega)$ tels que pour tous $q_1 \in \Lambda^0(\Omega)$, $\boldsymbol{v}_1 \in \Lambda^1(\Omega)$, $\boldsymbol{v}_2 \in \Lambda^2(\Omega)$, $q_2 \in \Lambda^3(\Omega)$,

$$\begin{aligned}
\int_\Omega \boldsymbol{u}_1 \cdot \mathbf{grad}\, q_1 &= -\int_\Omega g_1\, q_1\,, \\
\int_\Omega \boldsymbol{u}_2 \cdot (\mathbf{curl}\, \boldsymbol{v}_1) + \int_\Omega \mathbf{grad}\, p_1 \cdot \boldsymbol{v}_1 &= \int_\Omega (\boldsymbol{f}_2 + \boldsymbol{r}_1) \cdot \boldsymbol{v}_1\,, \\
\int_\Omega (\mathbf{curl}\, \boldsymbol{u}_1) \cdot \boldsymbol{v}_2 + \int_\Omega p_2 \operatorname{div} \boldsymbol{v}_2 &= \int_\Omega (\boldsymbol{f}_1 - \boldsymbol{r}_2) \cdot \boldsymbol{v}_2\,, \\
\int_\Omega \operatorname{div} \boldsymbol{u}_2\, q_2 &= \int_\Omega g_2\, q_2\,.
\end{aligned}$$

On retrouve un problème plus simple à résoudre connu sous le nom de problème de Hodge-Dirac [46]. Plus précisément pour f, $g \in L^2\Lambda(\Omega)$, en notant $f = \boldsymbol{r}_1 \oplus \boldsymbol{f}_1 \oplus g_2$ et $g = -g_1 \oplus \boldsymbol{f}_2 \oplus -\boldsymbol{r}_2$ on cherche $u = p_1 \oplus \boldsymbol{u}_1 \oplus \boldsymbol{u}_2 \oplus p_2$ tel que

$$\begin{cases} \mathrm{d}\, u = f, \\ \delta u = g. \end{cases}$$

Dans cet exemple, le calcul extérieur apparaît de façon réellement non triviale, en imposant de mélanger les différents espaces entre-eux. On obtient alors un schéma très robuste et l'on peut choisir arbitrairement d'avoir une divergence exacte ou un rotationnel exact. De plus les conditions de bord nécessaires et admissibles apparaissent très naturellement : il faut fixer, soit la composante normale, soit la composante tangentielle du champ recherché. Les conditions de compatibilité sur les sources sont également bien visibles.

Ce problème présente de très nombreuses applications dans des domaines variés aussi bien en électromagnétisme qu'en imagerie par ordinateur. Plus spécifiquement lié aux équations des fluides, ce problème apparaît dans une formulation des équations de Navier-Stokes consistant à résoudre en vorticité, ainsi que pour la détermination des fonctions des courants (nous nous en sommes servi par exemple dans l'application numérique 3.6.3) .

Dans le chapitre 3 nous nous sommes ensuite intéressés au cœur du problème avec la formulation de l'équation de Navier-Stokes en utilisant le calcul extérieur. Comme mentionné précédemment, nous utilisons l'identité de Lamb et la formulation du laplacien reposant sur le rotationnel. Nous considérons la vitesse comme étant une 2-forme et nous introduisons la vorticité $\boldsymbol{\omega}$ comme étant une 1-forme correspondant à la codifférentielle de $\boldsymbol{u}$. Nous prouvons des résultats de convergence ainsi que la conservation de certaines quantités. La conservation de ces quantités apparaît très simplement grâce à l'exactitude du complexe au niveau discret. Nous obtenons notamment la préservation exacte de la masse, la robustesse en pression (l'invariance de la vitesse par l'ajout d'un gradient aux forces extérieures), des estimations d'erreurs pour la vitesse indépendante de la pression, ainsi qu'une forme de conservation de l'énergie.

Ce schéma présente un réel champ d'applicabilité, par exemple pour les fluides tournants (voir chapitre 3.6.3). Cependant un problème fondamental de cette écriture est qu'elle ne permet pas d'imposer des conditions de sortie libre (sans contrainte). Il n'est pas encore clair si ce problème vient d'une mauvaise expression de la sortie libre ou que la traduction dans le complexe de De Rham nous échappe simplement.

Pour pallier ce problème, dans le chapitre 4 nous avons décidé de rechercher $\boldsymbol{u}$ avec une régularité $\boldsymbol{H}^1$ complète afin de contrôler toutes les composantes de sa trace. Bien que nous contrôlions les composantes normales et tangentielles aux interfaces, cela ne fait pas pour autant de $\boldsymbol{u}$ une 1-forme et simultanément une 2-forme comme nous l'avons évoqué dans le paragraphe 1.4.3. Concrètement nous revenons sur la formulation standard du laplacien (comme la divergence du gradient), mais nous considérons toujours $\boldsymbol{u}$ comme une 2-forme. Cela donne un nouveau complexe, appelé complexe de Stokes et différent du complexe de De Rham uniquement par la régularité des espaces :

$$\{0\} \xrightarrow{\ 0\ } H^2(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}^1(\mathrm{curl},\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\ 0\ } \{0\}.$$

Des discrétisations du complexe de Stokes existaient en dimension 2 depuis longtemps [31], cependant les analogues en dimension 3 étaient très rares et avec des contraintes importantes, notamment l'utilisation de polynômes de haut degré et des restrictions sur les conditions de bord (voir [41]). Nous avons donc créé un nouveau complexe de Stokes discret. En particulier nous avons réalisé un complexe entièrement discret sur des éléments polytopiques. Le terme entièrement discret signifie que les espaces discrets sont définis sur les éléments du maillage et ne sont pas des sous-espaces d'espaces continus. Ainsi le complexe ne vit pas dans un sous-espace de fonctions continues et les inconnues se situent sur des éléments de différentes dimensions (cellule, face, arête ou sommet). Ces dernières années, le sujet a été très actif et plusieurs complexes de Stokes sont parus dans la littérature au cours de la thèse (voir par exemple [69, 61, 63]). L'utilisation de complexes entièrement discrets a des inconvénients puisqu'ils héritent de l'exactitude du complexe adjoint plutôt que du complexe primal; dans notre cas (en considérant la vitesse comme une 2-forme) cela conduit à la perte de la robustesse en pression. Il est cependant possible d'avoir cette robustesse en considérant la vitesse comme une 1-forme (voir [70]). De plus la méthode bénéficie d'une grande flexibilité. Nous ne sommes pas restreints aux maillages simpliciaux ou cubiques, et pouvons utiliser des polyèdres arbitraires. De plus il préserve tout de même les autres avantages liés à l'utilisation d'un complexe tels que la grande simplification de l'analyse des schémas numériques ou la compatibilité avec d'autres systèmes d'équations comme en magnétohydrodynamique. Il préserve également le rotationnel, ce qui donne une invariance de la pression par l'ajout d'un champ solénoïdal aux forces extérieures avec une formulation **curl curl** du laplacien (voir le chapitre 4.1 pour plus de détails), c'est en quelque sorte la propriété duale de la robustesse en pression évoquée plus haut.

1.5.1 Implémentation.

Les travaux réalisés comportent également une partie numérique conséquente avec l'implémentation de toutes les méthodes proposées ci-dessus. Les codes sont librement disponibles en ligne sur le site GitHub à l'adresse `https://github.com/mlhanot`.

L'implémentation de la méthode pour le problème de Biot-Savart ainsi que celle pour les équations de Navier-Stokes (décrites respectivement dans les chapitres 2 et 3) on été réalisées à l'aide de la collection d'outils FEniCS (version 2019.1.0, voir `fenicsproject.org`). Le projet FEniCS est spécialisé dans la formulation et la résolution de méthodes d'éléments finis. En particulier les éléments finis d'algèbre extérieure décrits dans le tableau périodique [36] sont implémentés et utilisables de façon naturelle grâce au langage dédié « Unified Form Language ». On peut voir un exemple d'utilisation de ce langage dans le fragment de code 2.1. Nous avons également eu recours à la bibliothèque de solveurs linéaires et non linéaires PETSc `https://petsc.org`, ainsi qu'à la bibliothèque SLEPc `https://slepc.upv.es/` pour calculer les formes harmoniques discrètes.

L'implémentation du complexe de Stokes discret est plus lourde puisque le complexe relève d'une méthode hybride demandant de travailler simultanément avec des objets de plusieurs dimensions et traitant de maillages polyédriques quelconques. Elle a été réalisée grâce aux outils de la bibliothèque C++ HArDCore `https://github.com/jdroniou/HArDCore` à laquelle j'ai ajouté les espaces créés dans le complexe de Stokes `https://github.com/mlhanot/HArDCore3D-Stokes`.

Les résultat numériques obtenus sont discutés plus en détail à la fin de chaque article dans les chapitres 2.10, 3.6 et 4.7.

Chapitre 2

Numerical solution of the div-curl problem by finite element exterior calculus.

Abstract

We are interested in the numerical reconstruction of a vector field with prescribed divergence and curl in a general domain of $\mathbb{R}^3$ or $\mathbb{R}^2$, not necessarily contractible. To this aim, we introduce some basic concepts of finite element exterior calculus [50] and rely on recent results of P. Leopardi and A. Stern [46]. The goal of the paper is to take advantage of the links between usual vector calculus and exterior calculus and show the interest of the exterior calculus framework, without too much prior knowledge of the subject. We start by describing the method used for contractible domains and its implementation using the FEniCS library (see `fenicsproject.org`). We then address the problems encountered with non contractible domains and general boundary conditions and explain how to adapt the method to handle these cases. Finally we give some numerical results obtained with this method, in dimension 2 and 3.

2.1 Outline of the paper.

In electromagnetism or fluid mechanics, one often encounter the problem of reconstructing a vector field with prescribed divergence and curl. As is well-known, the Biot-Savart law allows the reconstruction of a solenoidal vector field u in the whole space $\mathbb{R}^d$ from its curl.

$$u(x,t) = \int_{\mathbb{R}^d} K(x,y) \times \operatorname{curl} u(y,t)\, dy$$

with

$$K(x,y) = \begin{cases} \frac{1}{4\pi}\frac{x-y}{|x-y|^3} & \text{if } d=3, \\ \frac{1}{2\pi}\frac{x-y}{|x-y|^2} & \text{if } d=2. \end{cases}$$

However it is a singular integral in the whole space and is not easily converted to a numerical algorithm, nor suitable in a bounded domain. The div-curl problem (2.1) has been addressed theoretically e.g. in [10] or [17]. The *numerical* computation of the solution in a bounded domain is less documented, see e.g. [42] for a finite element solution using divergence free elements and an coercive variational form. The purpose of this paper is to show that the framework of differential forms and exterior calculus greatly simplifies both the theory and the finite element solution. In section 2.2 we show how the div-curl problem is related to the classical Helmholtz decomposition. In section 2.3 we introduce the main tools of exterior calculus. In section 2.4 we give a natural weak formulation of the div-curl problem, introduced in [46], which is well posed. In section 2.5 we detail the mixed finite elements compatible with the weak formulation. The implementation within the unified form language [29] is sketched in section 2.6. The particular case of the 2D case is detailed in section 2.7. The case when the domain is not contractible is addressed in section 2.8. In section 2.9 we describe the cases of natural and essential boundary conditions. We close the paper with numerical tests in 2 and 3 dimensions in section 2.10.

2.2 Helmholtz decomposition

Let Ω be a bounded domain in $\mathbb{R}^d$, the div-curl problem consists in finding a vector field with a prescribed divergence and curl. For g a scalar field and f a vector field we seek u such that

$$\begin{aligned} \nabla \cdot u &= g \quad \text{in } \Omega, \\ \nabla \times u &= f \quad \text{in } \Omega. \end{aligned} \tag{2.1}$$

Of course, one must add boundary conditions and specify the regularity of these fields. The existence of a solution is not guaranteed, indeed some vector fields are not written as the curl of other vector fields. The uniqueness is also an issue, indeed depending on the domain topology there may exist non trivial fields with vanishing divergence and curl, the so called harmonic fields. Theses problem make the elaboration of a stable scheme for the numerical solution quite complicated, because one must make sure that the fields f and g are compatible and in some way filter out the harmonic fields.

Problem (2.1) is very close to the Helmholtz decomposition. The Helmholtz decomposition is central in vector calculus. In its classical formulation it states that any field of $\mathbb{R}^3$ which is sufficiently smooth and decreases sufficiently fast at infinity can be decomposed into the sum of a gradient and a curl. The problem is then to compute this decomposition. For a given vector field $\mathbf{F}$, find a vector potential $\mathbf{A}$ and a scalar potential ϕ such that

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}. \tag{2.2}$$

Although this is not equivalent to the equations (2.1), we can see the relation by taking $u = \mathbf{A}$ in (2.2) and $f = \mathbf{F}$.

Thus, the problem of the absence of compatibility of f in (2.1) can be solved by looking simultaneously for a vector potential u and a scalar potential ϕ as described by (2.2). The general idea is then to couple these problems to obtain a well-posed system.

The classical Helmholtz decomposition (2.2) assumes that functions are smooth in the whole space. For bounded domains, we have the following standard result, see e.g. [3] or chapter 9 of [4]. Let Ω be a bounded, simply-connected, Lipschitz domain in $\mathbb{R}^d$, for any $\mathbf{F} \in (L^2(\Omega))^d$ there exists $\phi \in H^1(\Omega)$ and $\mathbf{A} \in H(\text{curl}, \Omega)$ such that the decomposition (2.2) is valid. Of course, there is no uniqueness, since one can add constants to ϕ and any gradient field to $\mathbf{A}$. Prescribing boundary conditions, we get the L^2-orthogonal decompositions, proved e.g. in Arnold [50]:

$$(L^2(\Omega))^d = \nabla(H^1(\Omega)) \oplus \nabla \times (H_0(\text{curl}, \Omega)) \tag{2.3}$$

$$(L^2(\Omega))^d = \nabla(H_0^1(\Omega)) \oplus \nabla \times (H(\text{curl}, \Omega)). \tag{2.4}$$

We recall the standard definitions: $H_0^1(\Omega) = \{u \in L^2(\Omega);\ \nabla u \in L^2(\Omega)^d,\ u = 0 \text{ on } \partial\Omega\}$, $H_0(\text{curl}, \Omega)) = \{u \in L^2(\Omega)^d;\ \text{curl}(u) \in L^2(\Omega)^d,\ u \times n = 0 \text{ on } \partial\Omega\}$, n being the unit outer normal to the boundary $\partial\Omega$.

2.3 Exterior calculus.

Exterior calculus is based on differential forms, which are applications that at any point of the domain associate an alternating multilinear form. In particular the 0-forms are simply functions and the 1-forms can be seen as vector fields (by identifying linear forms with vectors, thanks to the usual inner product). These spaces are endowed with a natural inner product (built from the one on vectors), which can be integrated on the whole domain. This allows us to define Hilbert spaces, in particular we will note $L^2\Lambda^k(\Omega)$ the set of L^2-integrable k-forms. As alternating multilinear applications, it is possible to define an operation between a k-form α and an l-form β denoted by the wedge product $\wedge$ giving a $k + l$-form $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$. This operation is pointwise and allows to define a basis of the whole algebra from a basis of 1-forms. In dimension 3 a local basis is given in the table 2.1.

Space	$\Lambda^0(\Omega)$	$\Lambda^1(\Omega)$	$\Lambda^2(\Omega)$	$\Lambda^3(\Omega)$
Basis	1	dx, dy, dz	$dy \wedge dz, -dx \wedge dz, dx \wedge dy$	$dx \wedge dy \wedge dz$

Table 2.1: Basis of the exterior algebra on a domain of $\mathbb{R}^3$.

These forms are also equipped with the exterior derivative operator d. This operator acts globally on the exterior algebra $d : \bigoplus_{k=0} H\Lambda^k(\Omega) \to \bigoplus_{k=0} H\Lambda^k(\Omega)$ but it is often interesting to look at its action restricted to each space, where it will respect a notion of degree, specifically :

$$d : H\Lambda^k(\Omega) \to H\Lambda^{k+1}(\Omega) \ .$$

The operator d is defined as the usual differential on the 0-forms, by the property $d \circ d = 0$ and extended on the other forms degree by $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for a k-form α. This differential operator is of course not defined on all L^2-forms but only on a dense subset. As for the Sobolev spaces, we then consider the subset of L^2-forms such that their exterior derivative is again L^2. We note this set $H\Lambda(\Omega)$.

One can see a great similarity with vector calculus, for instance $d \circ d = 0$ corresponds to the well known identities curl(grad) = 0 and div(curl) = 0. This similarity is deep since there is a natural identification between vector fields and differential forms, the identification commutating with these differential operators. The identification is depicted by diagrams 2.2 for dimension 2 and 2.1 for dimension 3. We use the natural shortcut $dx \to x, dy \to y, dz \to z$ to mean that the one form $a\,dx + b\,dy + c\,dz$ is identified with the vector $a \cdot \mathbf{e_x} + b \cdot \mathbf{e_y} + c \cdot \mathbf{e_z} = (a, b, c)$. In the same way, for 2-forms $dy \wedge dz \to x$, $dz \wedge dx \to y$, $dx \wedge dy \to z$ means that $a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$ is identified with the vector (a, b, c), whereas for 3-forms $dx \wedge dy \wedge dz \to 1$ simply means that the 3 form $a\,dx \wedge dy \wedge dz$ is identified with the scalar a. In dimension 2, the same conventions apply. We can notice that there are two possible identifications in dimension 2. We will refer to the first one in which $\nabla\times$ appears as the curl identification and to the other one (in which $\nabla\cdot$ appears) as the divergence identification. We refer again to [50] for a thorough exposition of this so called ”proxy” identifications.

Advantages of using exterior calculus are twofold. First a lot of work has already been done for the discretization of these spaces and operator d (see [50], [24]) as we will see in Section 2.5. Second is that the previously described vector - differential form identifications makes the nature of our operators clearer. Instead of having three different operators ∇, $\nabla\cdot$ and $\nabla\times$ we see that they become after identification a single operator d applied to different spaces. This allows us to unify the different types of ”Helmholtz decomposition” (called Hodge decomposition in the context of exterior calculus) into a single one and allows us to understand how to find stable formulations of the problems (2.1) and (2.2).

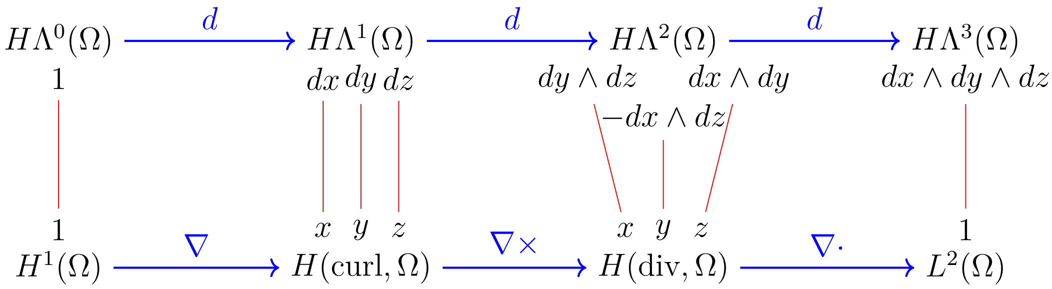


Figure 2.1: Identification between vectors and forms on a domain $\Omega \subset \mathbb{R}^3$.

Before stating this decomposition, let us introduce the adjoints of the operator d. Indeed, as we can see in the diagram 2.1, after identification the image of ∇ and that of $\nabla\times$ do not belong to the same space. This is perfectly normal because these two operators will not occur in the same decomposition, ∇ will occur in the decomposition of 1-forms and $\nabla\times$ in that of 2-forms. The operators completing these decompositions are the adjoints of these operators.

We define the codifferential operator δ as the adjoint operator of d. Since it is not defined on the whole $L^2\Lambda(\Omega)$, we note its domain $\dot{H}^\star\Lambda(\Omega)$. Explicitly for Ω a bounded domain of $\mathbb{R}^3$, the adjoint of $(\nabla, H^1(\Omega))$ is $(-\nabla\cdot, H_0(\text{div}, \Omega))$, the adjoint of $(\nabla\times, H(\text{curl}, \Omega))$ is $(\nabla\times, H_0(\text{curl}, \Omega))$ and the adjoint of $(\nabla\cdot, H(\text{div}, \Omega))$ is $(-\nabla, H^1_0(\Omega))$, [50] for details. Thus under the identification in fig. 2.1 we have

$$\begin{aligned} (-\nabla\cdot, H_0(\text{div}, \Omega)) &= (\delta, \dot{H}^\star\Lambda^1(\Omega)), \\ (\nabla\times, H_0(\text{curl}, \Omega)) &= (\delta, \dot{H}^\star\Lambda^2(\Omega)), \\ (-\nabla, H^1_0(\Omega)) &= (\delta, \dot{H}^\star\Lambda^3(\Omega)). \end{aligned} \tag{2.5}$$

Let Ω is a bounded, contractible, Lipschitz domain, to the two Helmholtz decompositions (2.3)-(2.4) correspond the two following Hodge decompositions, depending on the choice to identify $(L^2(\Omega))^3$ with $L^2\Lambda^1(\Omega)$ or with $L^2\Lambda^2(\Omega)$. They are respectively given by

$$L^2\Lambda^1(\Omega) = d(H\Lambda^0(\Omega)) \oplus \delta(\dot{H}^\star\Lambda^2(\Omega)), \tag{2.6}$$

$$L^2\Lambda^2(\Omega) = \delta(\dot{H}^\star\Lambda^3(\Omega)) \oplus d(H\Lambda^1(\Omega)). \tag{2.7}$$

Considering $\Lambda^4(\Omega) \equiv \{0\}$ we can also extend the decomposition to the 3-forms:

$$L^2\Lambda^3(\Omega) = d(H\Lambda^2(\Omega)).$$

This reflects the fact that $\nabla \cdot (H(\text{div}, \Omega)) = L^2(\Omega)$.

Finally, if we try to extend the formula to 0-forms we will encounter a problem. Indeed, the constant functions are not in the range of $(\nabla\cdot, H_0(\text{div}, \Omega))$. Moreover

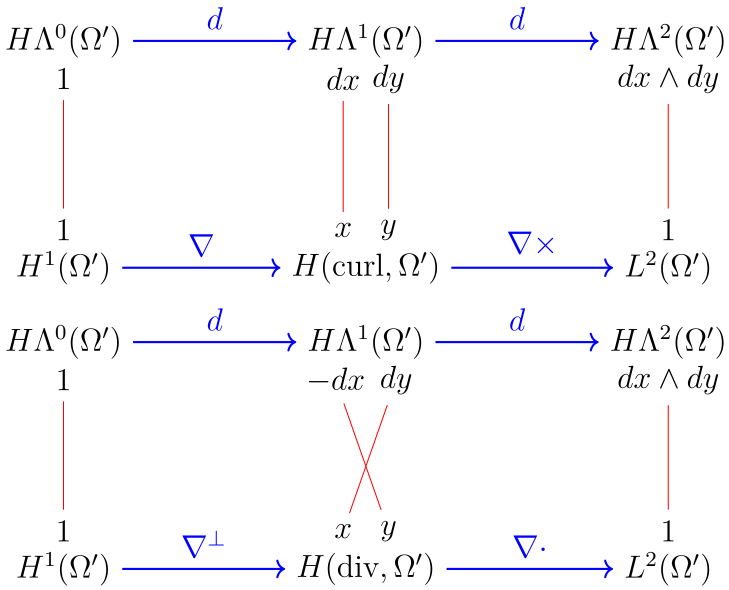


Figure 2.2: Two possibles identification between vectors and forms on a domain $\Omega' \subset \mathbb{R}^2$.

the constant functions are exactly the kernel of ∇ and are orthogonal to the range of $(\nabla\cdot, H_0(\mathrm{div}, \Omega))$, which is a generic fact that we will develop in Section 2.8 generalizing the algorithm to general domains. We will call such function harmonic (0-)forms and note their set $\mathfrak{H}^0 \subset L^2\Lambda^0(\Omega)$ (the set of constant 0-forms). The Hodge decomposition is then

$$L^2\Lambda^0(\Omega) = \delta(\dot{H}^\star\Lambda^1(\Omega)) \oplus \mathfrak{H}^0.$$

We can gather these four decompositions into a decomposition of the total space $L^2\Lambda(\Omega) = \bigoplus_{k=0}^{3} L^2\Lambda^k(\Omega)$ and on this domain, the Hodge decomposition is given by

$$L^2\Lambda(\Omega) = d(H\Lambda(\Omega)) \oplus \delta(\dot{H}^\star\Lambda(\Omega)) \oplus \mathfrak{H}^0. \tag{2.8}$$

2.4 Weak formulation.

The Hodge decomposition (2.8) gives the quasi invertibility of the operator $d+\delta : H\Lambda(\Omega) \cap \dot{H}^\star\Lambda(\Omega) \to L^2\Lambda(\Omega)$. We mean by that the following statement.

Theorem 2.1. *Let $f \in L^2\Lambda(\Omega)$, there exists $w \in H\Lambda(\Omega) \cap \dot{H}^\star\Lambda(\Omega)$ and $h \in \mathfrak{H}^0$ such that $f = (d+\delta)w + h$.*

Proof. Let f be in $L^2\Lambda(\Omega)$, from the Hodge decomposition 2.8, there exist $u \in H\Lambda(\Omega)$, $v \in \dot{H}^\star\Lambda(\Omega)$ and $h \in \mathfrak{H}^0$ such that

$$f = du + \delta v + h.$$

Let us apply the Hodge decomposition to u and v. We can write $u = dm + \delta p + q$ and $v = dr + \delta s + t$, where m, r belong to $H\Lambda(\Omega)$, p, s belong to $\dot{H}^\star\Lambda(\Omega)$ and q, t are in $\mathfrak{H}^0$. Now take $w = \delta p + dr$. Using $d^2 = \delta^2 = 0$ and the fact that q and t are harmonic forms, we can compute

$$(d+\delta)w = d\delta p + \delta dr = d(dm + \delta p + q) + \delta(dr + \delta s + t) = du + \delta v.$$

So we have proved that $f = (d+\delta)w + h$. Furthermore we have that $\delta p \in L^2\Lambda(\Omega)$ and $dm \in L^2\Lambda(\Omega)$, thus $w \in L^2\Lambda(\Omega)$. Now $dw = d\delta p = du$ belongs to $L^2\Lambda(\Omega)$ and $\delta w = \delta dr = \delta v$ belongs to $L^2\Lambda(\Omega)$, hence $w \in H\Lambda(\Omega) \cap \dot{H}^\star\Lambda(\Omega)$. □

This operator is called the Hodge-Dirac operator and is thoroughly studied in [46]. The non-invertibility of this operator comes from the space of harmonic forms $\mathfrak{H} = \mathfrak{H}^0$, where we omit the superscript from now on, being both in the kernel of the operator and orthogonal to its range. This can be circumvented by using a space orthogonal to the harmonic forms.

Hence the primal formulation of (2.1) becomes:
Given $f \in L^2\Lambda(\Omega) \cap \mathfrak{H}^\perp$, find $u \in H\Lambda(\Omega) \cap \dot{H}^\star\Lambda(\Omega) \cap \mathfrak{H}^\perp$ such that $\forall v \in H\Lambda(\Omega)$,

$$\langle du, v\rangle + \langle \delta u, v\rangle = \langle f, v\rangle. \tag{2.9}$$

By orthogonality arguments we can see that testing with $v \in \mathfrak{H}$ does not matter. Since d is adjoint to δ we can see that (2.9) is equivalent to:
Given $f \in L^2\Lambda(\Omega) \cap \mathfrak{H}^\perp$, find $u \in H\Lambda(\Omega) \cap \mathfrak{H}^\perp$ such that $\forall v \in H\Lambda(\Omega)$,

$$\langle du, v\rangle + \langle u, dv\rangle = \langle f, v\rangle. \tag{2.10}$$

We remove the condition of orthogonality $f \in \mathfrak{H}^\perp$ by introducing a new pair of variables. The problem becomes:
Given $f \in L^2\Lambda(\Omega)$, find $u \in H\Lambda(\Omega), p \in \mathfrak{H}$ such that $\forall v \in H\Lambda(\Omega), \forall q \in \mathfrak{H}$,

$$\langle du, v\rangle + \langle u, dv\rangle + \langle p, v\rangle = \langle f, v\rangle, \tag{2.11}$$

$$\langle u, q\rangle = 0. \tag{2.12}$$

We have added the component $\langle p, v\rangle$ to the equation (2.11), since by orthogonality we must have $p = P_{\mathfrak{H}} f$ (the orthogonal projection of f on $\mathfrak{H}$) and so we will effectively solve for $(d+\delta)u = f - P_{\mathfrak{H}} f$. Equation (2.12) ensures injectivity by imposing $P_{\mathfrak{H}} u = 0$. The additional terms can be seen as Lagrange multipliers making Problem (2.11)-(2.12) well-posed. We have the following result:

Theorem 2.2. *For any $f \in L^2\Lambda(\Omega)$, there is a unique $(u, p) \in H\Lambda(\Omega) \times \mathfrak{H}$ solution of (2.11)-(2.12). Moreover there exists $c > 0$ depending only on Ω such that $\|u\| + \|du\| + \|p\| \leq c\|f\|$.*

For the sake of completeness we give the proof, which follows closely the one given by Leopardi and Stern, [46, Theorem 6].

Proof. Let $u \in H\Lambda(\Omega)$ and $p \in \mathfrak{H}$, from the Hodge decomposition (2.8), there exist $m \in H\Lambda(\Omega)$, $n \in \dot{H}^\star\Lambda(\Omega)$ and $o \in \mathfrak{H}$ such that $u = dm + \delta n + o$. Applying once again the Hodge decomposition to m, there exist $r \in H\Lambda(\Omega)$, $s \in \dot{H}^\star\Lambda(\Omega)$ and $t \in \mathfrak{H}$ such that $m = dr + \delta s + t$. Take $v = d\delta n + \delta s + p$ and $q = o$ in (2.11)-(2.12). Noticing that $dv = d\delta s = dm$, $du = d\delta n$, using the Poincaré inequality $\|\delta n\| \leq c_p\|d\delta n\|$ (see for example [50, Theorem 4.6]), and using the orthogonality of the Hodge decomposition the equation reads:

$$\begin{aligned}
\langle du, v\rangle + \langle u, dv\rangle + \langle p, v\rangle + \langle u, q\rangle &= \langle d\delta n, d\delta n\rangle + \langle u, dm\rangle + \langle p, p\rangle + \langle u, o\rangle \\
&= \|d\delta n\|^2 + \|dm\|^2 + \|p\|^2 + \|o\| \\
&\geq \frac{1}{2}\|d\delta n\|^2 + \frac{1}{2c_p^2}\|\delta n\|^2 + \|dm\|^2 + \|p\|^2 + \|o\| \\
&\geq \frac{1}{2}\|du\|^2 + \min(1, \frac{1}{2c_p^2})\|u\|^2 + \|p\|^2.
\end{aligned} \tag{2.13}$$

Moreover we can bound the $H\Lambda$-norm of (v, q) by the $H\Lambda$-norm of (u, p) with a Poincaré inequality:

$$\begin{aligned}
\|v\|^2 + \|dv\|^2 + \|q\|^2 &= \|d\delta n\|^2 + \|\delta s\|^2 + \|p\|^2 + \|d\delta s\|^2 + \|o\|^2 \\
&\leq \|du\|^2 + c_p^2\|dm\|^2 + \|p\|^2 + \|dm\|^2 + \|o\|^2 \\
&\leq (1 + c_p^2)\|u\|^2 + \|du\|^2 + \|p\|^2.
\end{aligned} \tag{2.14}$$

By the symmetry of the formulation this is enough to conclude with the Babuška–Lax–Milgram theorem. □

Finally, let us translate this problem into the language of vector calculus in $\mathbb{R}^3$ (the $\mathbb{R}^2$ case being analogous with the appropriate definition of scalar and vector curl). First of all the unknowns are in fact a 4-tuple of fields, so we will write $u = (u_0, u_1, u_2, u_3)$, the subscript pertaining to the degree of the corresponding differential form. We will keep the notation $\mathfrak{H}$ for the space of harmonic forms which is simply a vector space of dimension 1 containing the constant functions. The problem is then written: Given $(f_0, f_1, f_2, f_3) \in L^2(\Omega) \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \times L^2(\Omega)$, find $u_0 \in H^1(\Omega)$, $u_1 \in H(\text{curl}, \Omega)$, $u_2 \in H(\text{div}, \Omega)$, $u_3 \in L^2(\Omega)$, $p \in \mathfrak{H}$ such that $\forall v_0 \in H^1(\Omega)$, $\forall v_1 \in H(\text{curl}, \Omega)$, $\forall v_2 \in H(\text{div}, \Omega)$, $\forall v_3 \in L^2(\Omega)$, $\forall q \in \mathfrak{H}$,

$$\begin{aligned} \langle u_1, \nabla v_0 \rangle + \langle p, v_0 \rangle &= \langle f_0, v_0 \rangle, \\ \langle u_2, \nabla \times v_1 \rangle + \langle \nabla u_0, v_1 \rangle &= \langle f_1, v_1 \rangle, \\ \langle u_3, \nabla \cdot v_2 \rangle + \langle \nabla \times u_1, v_2 \rangle &= \langle f_2, v_2 \rangle, \\ \langle \nabla \cdot u_2, v_3 \rangle &= \langle f_3, v_3 \rangle, \\ \langle u_0, q \rangle &= 0. \end{aligned} \tag{2.15}$$

This weak formulation was introduced in [46], where its wellposedness is also established, thanks to the standard finite element exterior calculus tools [50]. Natural boundary condition result from the weak formulation (2.15) hence a solution (u_0, u_1, u_2, u_3) satisfies $u_1 \in H_0(\text{div}, \Omega)$, $u_2 \in H_0(\text{curl}, \Omega)$ and $u_3 \in H_0^1(\Omega)$. Thus a solution of (2.15) satisfies

$$\begin{aligned} -\nabla \cdot u_1 + p &= f_0, \\ \nabla u_0 + \nabla \times u_2 &= f_1, \\ \nabla \times u_1 - \nabla u_3 &= f_2, \\ \nabla \cdot u_2 &= f_3, \end{aligned} \tag{2.16}$$

with the natural boundary conditions

$$\begin{aligned} u_1 \cdot n &= 0 \quad \text{on } \partial\Omega, \\ u_2 \times n &= 0 \quad \text{on } \partial\Omega, \\ u_3 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.17}$$

Solving the problem (2.15) simultaneously solves two div-curl problems (2.1), computes two Helmholtz decompositions (of f_1 and f_2) and finds two functions with prescribed gradient. Assuming that the f_i-functions are compatible (for the sake of simplicity only, otherwise the problem is solved for their orthogonal projection in the appropriate spaces) the problem div-curl (2.1) solved are:

$$\begin{aligned} \nabla \cdot u_1 &= -f_0, \\ \nabla \times u_1 &= f_2, \\ u_1 \cdot n &= 0 \text{ on } \partial\Omega, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \nabla \cdot u_2 &= f_3, \\ \nabla \times u_2 &= f_1, \\ u_2 \times n &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Remark 2.3. There are two differences between the two problems, first u_1 and u_2 do not follow the same boundary conditions, moreover as we shall see below in section 2.5, the discretization of the problem does not treat the differential and codifferential symmetrically. In particular, we have no error estimates for the convergence of the discrete codifferential $||\delta u - \delta_h u_h||_{L^2}$, see section 2.10.

2.5 Finite elements.

The design for finite elements suitable for exterior calculus saw substantial progress recently, with the seminal work of [24],[50]. For the Hodge-Dirac problem (as well as for the Hodge-Laplacian problem), these elements can be realized in a very generic way as shown in the periodic table of the finite elements [36].

The main properties of these elements are that they form a discrete subcomplex and admit bounded cochain projections. Being a discrete subcomplex means that the functions constructed on these elements belong to the domain of the exterior derivative (just as the functions are respectively included in H^1, $H(\text{curl})$, $H(\text{div})$, L^2) and that their derivative (respectively their gradient, curl and div) are included in the functions of the next element. This last property allows to have properties exactly verified at the discrete level (for example discrete fields with exactly zero divergence). Bounded cochain projections are projections from continuous space to discrete space commuting with the exterior derivative. The boundedness for different norms ensures stability and accurate estimation of the error. These projections exist mainly as theoretical tools and their calculation is never performed in the numerical scheme.

Remark 2.4. Although the discrete exterior derivative operator d_h is the same as the continuous d (more precisely its restriction on the discrete spaces), the discrete codifferential operator δ_h has little to do with the continuous operator δ. Indeed, it is the adjoint of the same operator but on a different space. This is why we have removed any occurrence of δ from the formulation (2.9).

The space of harmonic forms $\mathfrak{H}$ remains in this case the space of constant functions. Its determination can however be more complicated when using general domains or other boundary conditions. This problem is detailed in section 2.9.

Remark 2.5. The discrete spaces are then subspaces of continuous spaces and we use a conforming method. This will not always be the case for general domains where the space of discrete harmonic forms may be different from the continuous space, though they have the same dimensions, see [50].

Although other types of meshes such as quadrilaterals are possible [36], we focus on simplicial meshes. For each degree of forms, there are two families of piecewise polynomial elements indexed by their polynomial degree. Consider a simplicial mesh $\mathfrak{T}$ (and denote the cells of the mesh by $T \in \mathfrak{T}$) and a polynomial degree r. The first family is the complete space of polynomials, differing by the type of continuity desired at the interfaces, it is given by

$$\begin{aligned} P_r\Lambda^0(\mathfrak{T}) &= \{\omega \in H^1(\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r(T,\mathbb{R})\}, \\ P_r\Lambda^1(\mathfrak{T}) &= \{\omega \in H(\mathrm{curl},\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r(T,\mathbb{R}^3)\}, \\ P_r\Lambda^2(\mathfrak{T}) &= \{\omega \in H(\mathrm{div},\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r(T,\mathbb{R}^3)\}, \\ P_r\Lambda^3(\mathfrak{T}) &= \{\omega \in L^2(\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r(T,\mathbb{R})\}. \end{aligned} \tag{2.18}$$

The space $P_r(T,\mathbb{R}^k)$ denotes the set of polynomials of degree r, defined on the domain T with value in $\mathbb{R}^k$. The second family are the so-called *trimmed* space with fewer degrees of freedom, it follows the same continuity conditions at the interfaces as the first one but uses only a subset of the polynomials, the exact definition of this set can be found in [50], we will simply denote them here by $P_r^-\Lambda^k$. This second family is then given by

$$\begin{aligned} P_r^-\Lambda^0(\mathfrak{T}) &= \{\omega \in H^1(\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r^-\Lambda^0\}, \\ P_r^-\Lambda^1(\mathfrak{T}) &= \{\omega \in H(\mathrm{curl},\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r^-\Lambda^1\}, \\ P_r^-\Lambda^2(\mathfrak{T}) &= \{\omega \in H(\mathrm{div},\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r^-\Lambda^2\}, \\ P_r^-\Lambda^3(\mathfrak{T}) &= \{\omega \in L^2(\Omega),\ \forall T \in \mathfrak{T}, \omega_{|T} \in P_r^-\Lambda^3\}. \end{aligned} \tag{2.19}$$

We always have $P_r^-\Lambda^0(\mathfrak{T}) = P_r\Lambda^0(\mathfrak{T})$ and $P_r^-\Lambda^3(\mathfrak{T}) = P_{r-1}\Lambda^3(\mathfrak{T})$. The common names (as referred to in the table [36]) for these elements in $\mathbb{R}^3$ are

- Lagrange elements of degree r for $P_r^-\Lambda^0(\mathfrak{T})$,
- Nedelec's edge elements of the first kind for $P_r^-\Lambda^1(\mathfrak{T})$ and of the second kind for $P_r\Lambda^1(\mathfrak{T})$,
- Nedelec's face elements of the first kind for $P_r^-\Lambda^2(\mathfrak{T})$ and of the second kind for $P_r\Lambda^2(\mathfrak{T})$,
- discontinuous Galerkin for $P_r\Lambda^3(\mathfrak{T})$.

From these elements, we have to build a sequence of degree increasing forms. For each degree, we can take either the full polynomial element or the trimmed polynomial element. However, we must choose the appropriate polynomial degree, following a simple rule: if the next element is a complete polynomial we must go down one degree, if it is trimmed we keep the same polynomial degree. Thus for any degree r the sequences

$$P_r^-\Lambda^0(\mathfrak{T}) \to P_r^-\Lambda^1(\mathfrak{T}) \to P_r^-\Lambda^2(\mathfrak{T}) \to P_r^-\Lambda^3(\mathfrak{T}), \tag{2.20}$$

$$P_{r+3}\Lambda^0(\mathfrak{T}) \to P_{r+2}\Lambda^1(\mathfrak{T}) \to P_{r+1}\Lambda^2(\mathfrak{T}) \to P_r\Lambda^3(\mathfrak{T}) \tag{2.21}$$

are both correct.

The main difference between the two families comes from their approximation properties. We define the approximation error for a discrete space V_h embedded in a continuous space V and a function $u \in V$ by $E(u) = \inf_{v_h \in V_h} \|u - v_h\|$. Error estimates for schemes are often expressed in terms of these approximation errors. These errors depend on the size h of the cells in the mesh, and converge to 0 when h tends to 0. For these spaces we have in all cases an error estimate of the form $E(u) \leq C\, h^l \|u\|_{H^l}$ with C a constant independent of h and l the order of convergence. For complete spaces of degree r the order of convergence of the approximation error (of a sufficiently regular function) is $r+1$ and the order of convergence of its derivative is r. For trimmed spaces of degree r the order of convergence is r and the order of convergence of its derivative is also r.

Remark 2.6. To complete remark 2.3, we can see that in the discrete case another difference appears between the 1-forms and the 2-forms regarding the regularity of the solutions. According to whether we use Raviart-Thomas-Nedelec edge (resp. face) elements the discrete solution u_h belongs to $H(\mathrm{div}, \Omega)$ (resp. $H(\mathrm{curl}, \Omega)$). In the continuous case, the solution u belongs to both $H(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega)$.

Let $V_h^0 \to V_h^1 \to V_h^2 \to V_h^3$ be the chosen discrete space sequence and $\mathfrak{H}_h$ be the space of discrete harmonic forms, the discrete problem is then:
Given $(f_0, f_1, f_2, f_3) \in L^2(\Omega) \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \times L^2(\Omega)$ find $u_{h0} \in V_h^0$, $u_{h1} \in V_h^1$, $u_{h2} \in V_h^2$, $u_{h3} \in V_h^3$, $p_h \in \mathfrak{H}_h$ such that $\forall v_{h0} \in V_h^0$, $\forall v_{h1} \in V_h^1$, $\forall v_{h2} \in V_h^2$, $\forall v_{h3} \in V_h^3$, $\forall q_h \in \mathfrak{H}_h$,

$$\begin{aligned} \langle u_{h1}, \nabla v_{h0} \rangle + \langle p_h, v_{h0} \rangle &= \langle f_0, v_{h0} \rangle, \\ \langle u_{h2}, \nabla \times v_{h1} \rangle + \langle \nabla u_{h0}, v_{h1} \rangle &= \langle f_1, v_{h1} \rangle, \\ \langle u_{h3}, \nabla \cdot v_{h2} \rangle + \langle \nabla \times u_{h1}, v_{h2} \rangle &= \langle f_2, v_{h2} \rangle, \\ \langle \nabla \cdot u_{h2}, v_{h3} \rangle &= \langle f_3, v_{h3} \rangle, \\ \langle u_{h0}, q_h \rangle &= 0. \end{aligned} \tag{2.22}$$

The error estimates are calculated in [46]. Let K be the solution operator which takes $f \to u$ in (2.15) and π_h be the bounded cochain projection mentioned at the beginning of section 2.5. We define $\eta = \|(I - \pi_h)K\|$ and $\mu = \|(I - \pi_h)P_{\mathfrak{H}}\|$. In our case, both converge to 0, and in practice we can use their expressions computed in [24], which for discrete spaces using polynomial degrees r give $\eta = \mathcal{O}(h)$ and $\mu = \mathcal{O}(h^{r+1})$. The error estimate for u the continuous solution of (2.15) and u_h the discrete solution of (2.22) gives:

$$\begin{aligned} \|d(u - u_h)\| &\lesssim E(du), \\ \|u - u_h\| &\lesssim E(u) + \eta(E(du) + E(p)), \\ \|p - p_h\| &\lesssim E(p) + \mu E(du). \end{aligned} \tag{2.23}$$

Where the notation $A \lesssim B$ means that there is a constant C independent of u and h such that $A \leq CB$. The equations (2.23) include all the components of $u = (u_0, u_1, u_2, u_3)$.

2.6 Implementation.

With these elements, the implementation of the problem (2.22) in *unified form language (UFL)*, see [35], is straightforward:

```
degree = 2
elemf0 = FiniteElement('P', tetrahedron, degree)
elemf1 = FiniteElement('N1E', tetrahedron, degree)
elemf2 = FiniteElement('N1F', tetrahedron, degree)
elemf3 = FiniteElement('DG', tetrahedron, degree-1)
elemH = FiniteElement('Real', tetrahedron, 0)
W = MixedElement([elemf0,elemf1,elemf2,elemf3,elemH])

(u0,u1,u2,u3,uh) = TrialFunctions(W)
(v0,v1,v2,v3,vh) = TestFunctions(W)
a1 = (dot(grad(u0),v1) + dot(curl(u1),v2) + div(u2)*v3)*dx
a2 = (dot(u1,grad(v0)) + dot(u2,curl(v1)) + u3*div(v2))*dx
ah = uh*v0*dx + u0*vh*dx
a = a1 + a2 + ah
```

Listing 2.1: Implementation of the variational formulation in ufl.

Our FEniCS codes and tests are available through the GitHub repository `https://github.com/mlhanot/divcurl_solver`.

Being a mixed finite element scheme, the assembled linear system is semidefinite, which for very large number of degrees of freedom could be an issue. However for contractible domains we obtained reliable results on a standard computer (16 GB RAM) with direct solvers up to several millions of degrees of freedom. For non contractible domains, the main workload is generated by the computations of a linear basis of the harmonic forms. Indeed this amounts to computing a basis of the null space of a matrix, which is trickier than solving a regular linear system. Note that the basis can be computed with more efficient, tree-based methods, such as the Dlotko-Specogna algorithm.

2.7 Problem in two dimensions.

The exterior calculus framework is unified w.r.t. dimensions, hence the formulation (2.11)-(2.12) remains perfectly valid without any change (except that one needs Ω to be domain of $\mathbb{R}^2$ instead of $\mathbb{R}^3$).

The expression in the vector calculus formalism is however quite different, in particular there are two possibilities of identification. The two corresponding problems are then (respectively for the curl identification and the div identification): Given $(f_0, f_1, f_2) \in L^2(\Omega) \times (L^2(\Omega))^2 \times L^2(\Omega)$, find $u_0 \in H^1(\Omega)$, $u_1 \in H(\text{curl}, \Omega)$, $u_2 \in L^2(\Omega)$, $p \in \mathfrak{H}$ such that $\forall v_0 \in H^1(\Omega)$, $\forall v_1 \in H(\text{curl}, \Omega)$, $\forall v_2 \in L^2(\Omega)$, $\forall q \in \mathfrak{H}$,

$$\begin{aligned} \langle u_1, \nabla v_0 \rangle + \langle p, v_0 \rangle &= \langle f_0, v_0 \rangle, \\ \langle u_2, \nabla \times v_1 \rangle + \langle \nabla u_0, v_1 \rangle &= \langle f_1, v_1 \rangle, \\ \langle \nabla \times u_1, v_2 \rangle &= \langle f_2, v_2 \rangle, \\ \langle u_0, q \rangle &= 0. \end{aligned} \tag{2.24}$$

Given $(f_0, f_1, f_2) \in L^2(\Omega) \times (L^2(\Omega))^2 \times L^2(\Omega)$, find $u_0 \in H^1(\Omega)$, $u_1 \in H(\text{div}, \Omega)$, $u_2 \in L^2(\Omega)$, $p \in \mathfrak{H}$ such that $\forall v_0 \in H^1(\Omega)$, $\forall v_1 \in H(\text{div}, \Omega)$, $\forall v_2 \in L^2(\Omega)$, $\forall q \in \mathfrak{H}$,

$$\begin{aligned} \langle u_1, \nabla^{\perp} v_0 \rangle + \langle p, v_0 \rangle &= \langle f_0, v_0 \rangle, \\ \langle u_2, \nabla \cdot v_1 \rangle + \langle \nabla^{\perp} u_0, v_1 \rangle &= \langle f_1, v_1 \rangle, \\ \langle \nabla \cdot u_1, v_2 \rangle &= \langle f_2, v_2 \rangle, \\ \langle u_0, q \rangle &= 0. \end{aligned} \tag{2.25}$$

The difference between the two is then the same as in the remark 2.3, namely the boundary conditions and the preferred operator.

The choice of finite elements must also be appropriate, although similar to the 3-dimensional case, one must be careful about the vector element chosen. The appropriate elements are (using the same semantics as for (2.18) and (2.19)):

- Lagrange elements for $P_r^- \Lambda^0(\mathfrak{T})$,
- Raviart-Thomas face (or edge) elements for $P_r^- \Lambda^1(\mathfrak{T})$,
- Brezzi-Doublas-Marini face (or edge) elements for $P_r \Lambda^1(\mathfrak{T})$,
- discontinuous Galerkin for $P_r \Lambda^2(\mathfrak{T})$.

We must use edge elements for the problem (2.24) and face elements for the problem (2.25).

Remark 2.7. The problem (2.25) is in fact simply the problem (2.24) to which we apply a quarter rotation on the vector space. This means that we solve the problem in the space of differential forms and only then apply the identification of Fig. 2.2 to come back to the vector fields.

2.8 Non contractible domain and harmonic forms.

When the domain is no longer contractible, the Hodge decomposition as given in (2.8) is no longer valid. As described in section 2.3 the problem comes from the appearing of harmonic forms, i.e. elements in the kernel of the Hodge-Dirac operator (equivalently in the kernel of the Hodge-Laplacian or fields f such that $\nabla \cdot f = 0$ and $\nabla \times f = 0$).

This is not really an issue because we already had harmonic forms in (2.9) and we can treat them in the same way. However this poses the problem of determining the space of harmonic forms The theoretical aspect of the problem is solved thanks to the famous theorem of De Rham giving the isomorphism between the harmonic forms and the cohomology of the cochain complex. This cohomology has a very strong geometrical interpretation, its dimension is given by the Betti numbers. Thus in dimension 2 the number of harmonic 0-forms is the number of connected components of the domain, the number of harmonic 1-forms is the number of holes of the domain and there are no harmonic 2-forms. In dimension 3 the number of harmonic 0-forms is still the number of connected components, the number of harmonic 1-forms corresponds to the number of tunnels and the number of harmonic 2-forms corresponds to the number of vacuum bubbles, there is no harmonic 3-forms. This is illustrated in figure 2.3 showing the two harmonic 1-forms on a disk with two holes, and figures 2.4 and 2.5 showing the two harmonic 1-forms on a hollow torus.

Another important theorem gives the isomorphism between discrete and continuous harmonic forms (see [50], chapter 5 and 7.6), so the dimension of the space of harmonic forms does not depend on the discretization or the elements chosen.

However the actual search for these forms is much more complicated. The isomorphisms only give their dimension number, and a heuristic idea of their shapes. To compute them, we will start from their definition as kernel of the Hodge-Dirac operator, which corresponds exactly to the kernel of the assembled matrix of the system (see [24]). The problem of determining a basis of harmonic forms becomes in practice a problem of finding kernels of matrices. The dimension of these kernels is a useful information for many algorithms (for example to search for the smallest eigenvalues) and the idea of the solution shapes can give a good initial guess. However, the problem remains intricate because of the size of the linear systems.

In our numerical computations with FEniCS, we have achieved the best results with the numerical algebra library SLEPc, see `https://slepc.upv.es/`.

Once a basis of harmonic forms is determined, it is enough to add them to the space $\mathfrak{H}$ in the scheme. The proofs are done in this general framework and still give the right estimates.

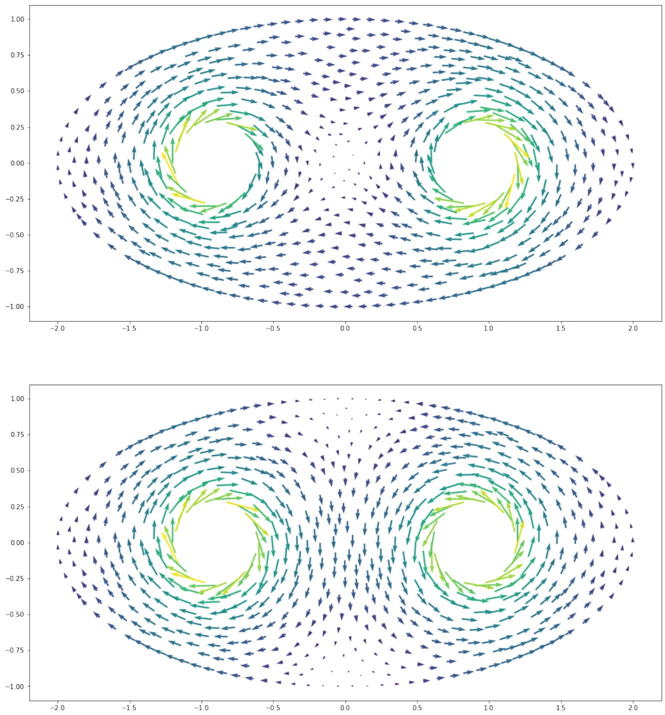


Figure 2.3: Harmonic 1-forms on a surface with two holes.

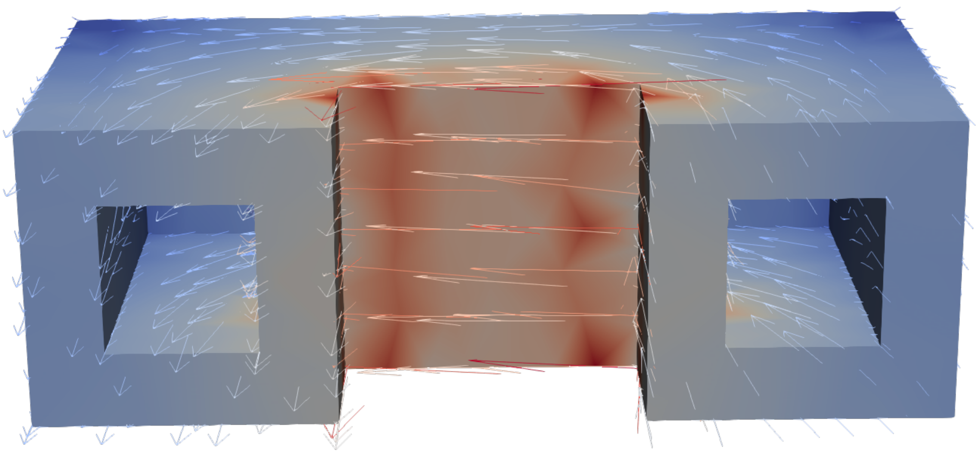


Figure 2.4: First harmonic 1-form on a 3D hollow torus (cross-section for visualisation).

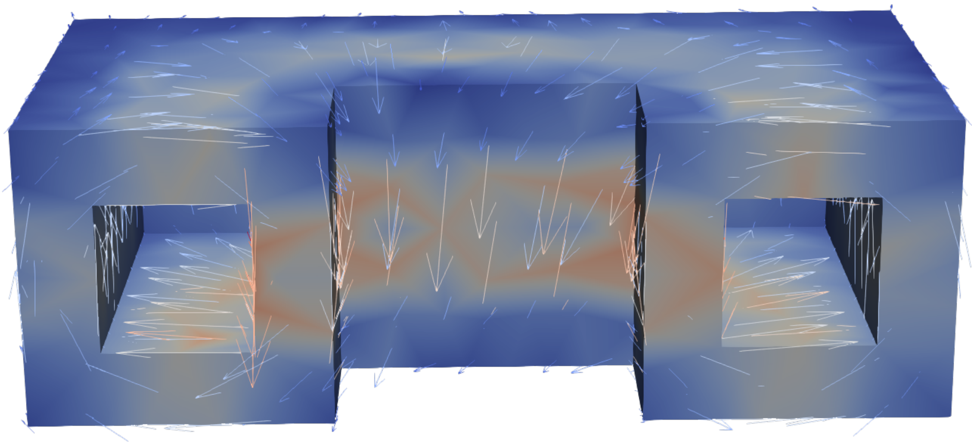


Figure 2.5: Second harmonic 1-form on a 3D hollow torus (cross-section for visualisation).

2.9 Boundary conditions.

So far, we have not imposed any essential (Dirichlet) conditions on our spaces, so natural conditions have emerged. Although these conditions are sufficient to ensure wellposedness, they fix the degree of forms in which we look for our solution and we have noticed in the remarks 2.3 and 2.6 that these choices have an impact on the convergence and the regularity of our solutions. Specifically, natural boundary conditions $u \times n = 0$ (resp. $u \cdot n = 0$) imposes u to be a 2-form, i.e. $u \in H(\mathrm{div}, \Omega)$ (resp. u to be a 1-form, i.e. $H(\mathrm{curl}, \Omega)$). If we wish that u_h belongs to the space which does not correspond to the natural condition obtained, a simple way is to apply homogeneous Dirichlet conditions to all spaces. The sequence then becomes

$$H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} H_0(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} L^2(\Omega). \tag{2.26}$$

From a theoretical point of view, enforcing Dirichlet conditions on all spaces does not pose any problem, we just end up with another complex, dual of the first one and all the theorems still work with one difference: the number (dimension) of harmonic forms are inverted, so under these conditions there are no 0 harmonic forms and as many 3 harmonic forms (in dimension 3) as there are connected components. One must adjust the $\mathfrak{H}$ space in the formulation, so replace $\langle p, v_0 \rangle$ by $\langle p, v_3 \rangle$ in (2.15).

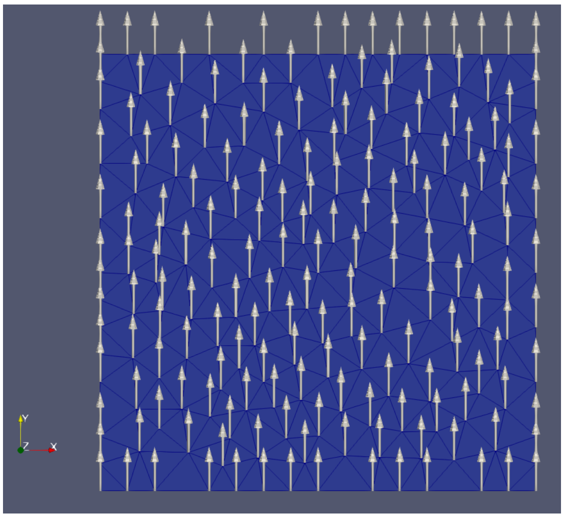


Figure 2.6: Harmonic 1-form on a contractible domain with mixed boundary conditions.

The situation becomes more complicated when mixed conditions are applied (natural on some faces, essential on others). New harmonic forms can then appear even for simple domains as illustrated in figure 2.6.

2.10 Numerical application.

To conclude we present some results computed with the weak form (2.22). In dimension 2 we take for reference function (2.27) on the unit square $(0,1)^2$, tailored to accommodate the boundary conditions, to have a non-trivial divergence and curl without being symmetric. In dimension 3 we take for reference function (2.28) on the unit cube $(0,1)^3$. The test with non-trivial harmonic forms is delicate, indeed the reference function will not be orthogonal to these forms and these forms are not known for most of the meshes. Thus we computed the rate of convergence in dimension 2 only, using periodic conditions on the edges $(x = 0)$ and $(x = 1)$ of the unit square with vanishing tangential components on the other edges because the harmonic forms are explicitly known in this case, see Fig. 2.6.

$$u = \begin{pmatrix} \sin(3\pi x)\cos(\pi y) \\ \sin(\pi y)\cos(2\pi x) \end{pmatrix} \tag{2.27}$$

$$u = \begin{pmatrix} \sin(3\pi x)\cos(\pi y)z \\ \sin(\pi y)\cos(2\pi x) + z \\ \sin(\pi z)\cos(3\pi x)\cos(\pi y) \end{pmatrix} \tag{2.28}$$

h	$\|u - u_h\|$	rate	$\|\nabla \cdot (u - u_h)\|$	rate	$\|\nabla \times (u - Pu_h)\|$	rate
0.1414	0.1753	—	1.1268	—	2.8449	—
0.0707	0.0854	1.03	0.5660	0.99	4.8353	-0.76
0.0353	0.0427	0.99	0.2833	0.99	4.7347	0.03
0.0176	0.0213	1.00	0.1416	0.99	4.8947	-0.04
0.0088	0.0107	0.99	0.0708	0.99	4.3588	0.16
0.0044	0.0053	0.99	0.0354	0.99	4.0170	0.11
0.0022	0.0026	1.01	0.0177	0.99	4.7146	-0.23

Table 2.2: Convergence rates for the sequence (2.20) of degree 1 in 2-dimension with the divergence identification.

We report the rate of convergence of the discrete solution towards the analytical solution, the rate of convergence of its differential (divergence or curl as the case may be) and the rate of convergence of its codifferential. The latter is calculated in two steps, first we make the orthogonal projection P of the function on an appropriate space (if the function has been defined on the face elements we project it on the corresponding edge elements and inversely). Then we compute its codifferential. Although there is no theoretical result on this convergence to our knowledge, we can observe in the last columns of the tables a convergence rate for the codifferential but with a loss of one degree with respect to the approximation property of the space of u_h. Of course as reported in tables 2.2, 2.6, 2.8 and 2.10, when degree one polynomials are used no convergence of the codifferentials can be expected.

Using a contractible domain and the sequence of finite elements (2.20), we can see the convergence rates in dimension 2 for the divergence identification for polynomials of degree 1 and 2 in tables 2.2 and 2.3, for the curl identification in tables 2.6 and 2.7. In dimension 3 we can see the convergence rates for 1-forms in tables 2.8 and 2.9, for 2-forms in tables 2.10 and 2.11. We can see the convergence rates with the sequence of elements (2.21) of degree $r = 0$ and $r = 1$ in dimension 2 with the divergence identification in tables 2.4 and 2.5. Finally, we can see the rate of convergence when harmonic 1-forms are present in dimension 2 with the divergence identification using the sequence (2.21) of polynomial degree 2 in the table 2.12. We notice in tables 2.4 and 2.5 that using *different* polynomials degrees like in equation (2.21) we have no loss of convergence for the codifferential. We have no theoretical proof of this fact.

h	$\|u-u_h\|$	rate	$\|\nabla\cdot(u-u_h)\|$	rate	$\|\nabla\times(u-Pu_h)\|$	rate
0.1414	0.04518	—	0.43621	—	1.09441	—
0.0707	0.01084	2.05	0.11165	1.96	0.50165	1.12
0.0353	0.00272	1.99	0.02806	1.99	0.25539	0.97
0.0176	0.00068	1.99	0.00702	1.99	0.12753	1.00
0.0088	0.00017	1.99	0.00175	1.99	0.06557	0.95
0.0044	0.00004	1.99	0.00043	1.99	0.03346	0.97

Table 2.3: Convergence rates for the sequence (2.20) of degree 2 in 2-dimension with the divergence identification.

h	$\|u-u_h\|$	rate	$\|\nabla\cdot(u-u_h)\|$	rate	$\|\nabla\times(u-Pu_h)\|$	rate
0.1414	0.027239	—	1.126863	—	0.909705	—
0.0707	0.006622	2.04	0.566014	0.99	0.401096	1.18
0.0353	0.001653	2.00	0.283319	0.99	0.202041	0.98
0.0176	0.000413	2.00	0.141698	0.99	0.101697	0.99
0.0088	0.000103	1.99	0.070854	0.99	0.050725	1.00
0.0044	0.000026	1.99	0.035427	0.99	0.025886	0.97
0.0022	0.000006	2.00	0.017713	0.99	0.012923	1.00

Table 2.4: Convergence rates for the sequence (2.21) of degree 2-1-0 in 2-dimension with the divergence identification.

h	$\|u-u_h\|$	rate	$\|\nabla\cdot(u-u_h)\|$	rate	$\|\nabla\times(u-Pu_h)\|$	rate
0.1414	0.041798	—	0.436218	—	0.092080	—
0.0707	0.009947	2.07	0.111658	1.96	0.021575	2.09
0.0353	0.002500	1.99	0.028061	1.99	0.005424	1.99
0.0176	0.000626	1.99	0.007024	1.99	0.001352	2.00
0.0088	0.000156	1.99	0.001756	1.99	0.000344	1.97
0.0044	0.000039	1.99	0.000439	1.99	0.000087	1.97

Table 2.5: Convergence rates for the sequence (2.21) of degree 3-2-1 in 2-dimension with the divergence identification.

h	$\|u-u_h\|$	rate	$\|\nabla\times(u-u_h)\|$	rate	$\|\nabla\cdot(u-Pu_h)\|$	rate
0.1414	0.1754	—	0.6344	—	3.9210	—
0.0707	0.0832	1.07	0.3182	0.99	3.8695	0.01
0.0353	0.0414	1.00	0.1592	0.99	3.6632	0.07
0.0176	0.0207	1.00	0.0796	0.99	3.5675	0.38
0.0088	0.0104	0.99	0.0398	0.99	3.3669	0.08
0.0044	0.0052	0.98	0.0199	0.99	2.9584	0.18
0.0022	0.0026	1.01	0.0099	0.99	3.1698	-0.09

Table 2.6: Convergence rates for the sequence (2.20) of degree 1 in 2-dimension with the curl identification.

h	$\|u-u_h\|$	rate	$\|\nabla\times(u-u_h)\|$	rate	$\|\nabla\cdot(u-Pu_h)\|$	rate
0.1414	0.029911	—	0.207629	—	1.375620	—
0.0707	0.007369	2.02	0.052777	1.97	0.671374	1.03
0.0353	0.001851	1.99	0.013249	1.99	0.342535	0.97
0.0176	0.000463	1.99	0.003315	1.99	0.171992	0.99
0.0088	0.000116	1.99	0.000829	1.99	0.086888	0.98
0.0044	0.000029	1.99	0.000207	1.99	0.044053	0.97

Table 2.7: Convergence rates for the sequence (2.20) of degree 2 in 2-dimension with the curl identification.

h	$\|u-u_h\|$	rate	$\|\nabla\times(u-u_h)\|$	rate	$\|\nabla\cdot(u-Pu_h)\|$	rate
0.3464	0.3128	—	2.2284	—	4.1974	—
0.1732	0.1590	0.97	1.1579	0.94	4.2920	-0.03
0.0866	0.0790	1.00	0.5852	0.98	4.1096	0.03
0.0044	0.0393	1.00	0.2934	0.99	3.9851	0.04

Table 2.8: Convergence rates for the sequence (2.20) of degree 1 for 1-forms in 3-dimension.

h	$\|u-u_h\|$	rate	$\|\nabla\times(u-u_h)\|$	rate	$\|\nabla\cdot(u-Pu_h)\|$	rate
0.3464	0.0598	—	0.5617	—	1.9609	—
0.1732	0.0151	1.97	0.1476	1.92	0.9729	1.01
0.0866	0.0038	1.99	0.0373	1.98	0.4924	0.98

Table 2.9: Convergence rates for the sequence (2.20) of degree 2 for 1-forms in 3-dimension.

h	$\|u - u_h\|$	rate	$\|\nabla \cdot (u - u_h)\|$	rate	$\|\nabla \times (u - Pu_h)\|$	rate
0.3464	0.2949	—	1.0490	—	4.4544	—
0.1732	0.1439	1.03	0.5307	0.98	3.8759	0.20
0.0866	0.0722	0.99	0.2663	0.99	3.9021	-0.00
0.0044	0.0361	0.99	0.1333	0.99	3.9106	-0.00

Table 2.10: Convergence rates for the sequence (2.20) of degree 1 for 2-forms in 3-dimension.

h	$\|u - u_h\|$	rate	$\|\nabla \cdot (u - u_h)\|$	rate	$\|\nabla \times (u - Pu_h)\|$	rate
0.3464	0.1083	—	0.8580	—	1.9663	—
0.1732	0.02649	2.03	0.2301	1.89	0.9586	1.03
0.0866	0.0067	1.97	0.0587	1.97	0.4971	0.94

Table 2.11: Convergence rates for the sequence (2.20) of degree 2 for 2-forms in 3-dimension.

h	$\|u - u_h\|$	rate	$\|\nabla \cdot (u - u_h)\|$	rate	$\|\nabla \times (u - Pu_h)\|$	rate
0.1414	0.04518	—	0.43621	—	1.09441	—
0.0707	0.01084	2.05	0.11165	1.96	0.50165	1.12
0.0353	0.00272	1.99	0.02806	1.99	0.25539	0.97
0.0176	0.00068	1.99	0.00702	1.99	0.12753	1.00
0.0088	0.00017	1.99	0.00175	1.99	0.06557	0.95

Table 2.12: Convergence rates for the sequence (2.20) of degree 2 in 2-dimension with the divergence identification on a non contractible domain.

Chapitre 3

An arbitrary order and pointwise divergence-free finite element scheme for the incompressible 3D Navier-Stokes equations.

Abstract

In this paper we discretize the incompressible Navier-Stokes equations in the framework of finite element exterior calculus. We make use of the Lamb identity to rewrite the equations into a vorticity-velocity-pressure form which fits into the de Rham complex of minimal regularity. We propose a discretization on a large class of finite elements, including arbitrary order polynomial spaces readily available in many libraries. The main advantage of this discretization is that the divergence of the fluid velocity is pointwise zero at the discrete level. This exactness ensures pressure robustness. We focus the analysis on a class of linearized equations for which we prove well-posedness and provide a priori error estimates. The results are validated with numerical simulations.

3.1 Introduction

We are interested in numerical schemes preserving the algebraic structure of the incompressible Navier-Stokes equations. Recently much work has been done to design structure preserving methods, but while the construction of such methods was found early on in two dimensions, the three-dimensional case remained difficult and the introduction of the finite element exterior calculus brought a significant breakthrough. An excellent review is given by V. John et al. [48]. The general idea taken from the finite element exterior calculus is to use a subcomplex of the De Rham complex. There are well-known discrete counterparts of this complex with minimal regularity, however the discretization of smoother variants is still an active topic usually leading to shape functions of high degree, see e.g. [41]. We chose to use the complex with minimal regularity, as this is often done for electromagnetism or recently for magnetohydrodynamics (see [66]).

The main difference from usual schemes lies in the regularity of the velocity field since we only require it to be in $H(\text{div})$ and in the discrete adjoint of $H(\text{curl})$. Although the continuous space regularity is the same as the usual one, since the adjoint of $(\text{curl}, H(\text{curl}))$ is $(\text{curl}, H_0(\text{curl}))$, and the velocity is sought in $H(\text{div}) \cap H_0(\text{curl}) \subset H^1$ (for a smooth enough domain, see [3, Part 3.2]). This does not hold (in general) in the discrete case since for $V_h \subset H(\text{curl})$ and (curl^*, V_h^*) the adjoint of (curl, V_h) we no longer have $V_h^* \subset H(\text{curl})$. This has a fundamental impact both from the philosophical and practical point of view. In practice $v \in H(\text{div})$ (resp. $v \in H(\text{curl})$) does not impose continuity of the tangential (resp. normal) components on faces. This suggests that we will not have any degree of freedom corresponding to these and lack any way to set them in a Dirichlet boundary condition. This means that the normal and the tangential part of the boundary condition must be treated in two different ways. It makes more sense in the exterior algebra and means that the fluid velocity is really sought as a 2-form (mostly defined by its flux across cell boundaries) which happens to be regular enough to also be in the domain of the exterior derivative adjoint.

Let us summarize the main idea of the algorithm. In order to preserve the divergence free constraint, we have to consider u as a 2-form, which can be discretized by face elements. Then it is not straightforward to discretize the Laplacian in the usual way $\langle \nabla u, \nabla v \rangle$ because ∇u is not a natural quantity for a 2-form. Our simple trick is to use $\nabla \cdot u = 0$ to rewrite the Laplacian:

$$\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = -\nabla \times (\nabla \times u). \tag{3.1}$$

Let Ω be a bounded domain of $\mathbb{R}^3$ and $T > 0$, we recall the Navier-Stokes equations:

$$\begin{aligned} u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T) \end{aligned} \tag{3.2}$$

together with some boundary and initial conditions, where u is the velocity of the fluid, p the pressure, ν the kinematic viscosity and f an external force.
Using the Lamb identity $(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla(u \cdot u)$ and Equation (3.1), we get the following formulation:

$$\begin{aligned} u_t + (\nabla \times u) \times u + \nu \nabla \times (\nabla \times u) + \nabla P = f \text{ on } \Omega \times (0,T), \\ \nabla \cdot u = 0 \text{ on } \Omega \times (0,T) \end{aligned} \tag{3.3}$$

where $P := p + \frac{1}{2} u \cdot u$ is the Bernoulli pressure.

Since u is a 2-form it is not natural to take $\nabla \times u$ (and it is unadvisable for reasons detailed in Remark 3.6). Therefore, we introduce an auxiliary variable $\omega = \nabla \times u$ (namely the vorticity) and work with a mixed problem. This is known as the vorticity-velocity-pressure formulation and was considered by many others (see [27, 21, 33, 13, 55]). The finite element exterior calculus framework is very flexible because it allows us to work in abstract spaces which can be discretized easily provided that some exactness properties are fulfilled. Therefore, we shall use a generic name for our spaces, here $V^1 \times V^2 \times V^3$ for the continuous spaces and $V_h^1 \times V_h^2 \times V_h^3$ for the discrete spaces (indexed by the mesh size h). Stating the exact requirements for these spaces requires introducing some concepts and notation, hence for the sake of readability we postpone the definition to Section 3.2.1. Typically a valid choice is to take $V^1 = H(\mathrm{curl}, \Omega)$, $V^2 = H(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$. We also need another space $\mathfrak{H}^3 = (\nabla \cdot V^2)^{\perp_{V^3}}$, the orthogonal complement of the range of (div, V^2) in V^3. This is a vector space that depends on the couple $V^2 \times V^3$, and which is typically of small dimension (i.e. 0 or 1).

Remark 3.1. The choice of boundary conditions is encoded in the choice of $V^1 \times V^2 \times V^3$. More details are given later in Section 3.2.2.

An example of discrete in time, mixed and linearized weak formulation is: Given $f^n \in L^2(\Omega)$, find $(\omega^n, u^n, p^n, \phi^n) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall (\tau, v, q, \chi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$,

$$\langle \omega^n, \tau \rangle - \langle u^n, \nabla \times \tau \rangle = 0, \tag{3.4a}$$

$$\langle \frac{1}{\delta t} u^n, v \rangle + \langle \nu \nabla \times \omega^n + \theta \omega^n \times u^{n-1} + (1-\theta)\omega^{n-1} \times u^n, v \rangle$$
$$- \langle p^n, \nabla \cdot v \rangle = \langle \frac{1}{\delta t} u^{n-1}, v \rangle + \langle f^n, v \rangle, \tag{3.4b}$$

$$\langle \nabla \cdot u^n + \phi^n, q \rangle = 0, \tag{3.4c}$$

$$\langle p^n, \chi \rangle = 0. \tag{3.4d}$$

Here $n \geq 1$ is the index of a linearly implicit time discretization and $\theta \in [0, 1]$ is an arbitrary parameter. In the following we take $\theta = \frac{1}{2}$. Let us look at a specific example: If we take $V^1 = H(\mathrm{curl}, \Omega)$, $V^2 = H(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$ then we must have $\mathfrak{H}^3 = \{0\}$. Equation (3.4a) is equivalent to $\omega^n = \nabla \times u^n$ and $u^n \in H_0(\mathrm{curl}, \Omega)$, (3.4b) is equivalent to $\frac{u^n - u^{n-1}}{\delta t} + \nu \nabla \times (\nabla \times u^n) + \frac{1}{2}(\omega^n \times u^{n-1} + \omega^{n-1} \times u^n) + \nabla p^n = f^n$ and $p^n \in H_0^1$, (3.4c) is equivalent to $\nabla \cdot u^n = 0$, and (3.4d) is trivial here (since $\mathfrak{H}^3 = \{0\}$). This formulation is similar to the one studied by Anaya et al. [55]. The main difference is that our formulation is studied in the framework of finite element exterior calculus and for arbitrary low order perturbations (see Equation (3.5) below). In particular the abstraction made on discrete spaces allows using any discrete subcomplex (as defined in Section 3.2.3). Two families are given as examples in Section 3.2.1 but more exist on different kind of meshes (e.g. the cubical elements [50, Chapter 7.7]). The construction of such families is still an active topic and the independence over the choice of discrete subcomplex is a great feature of finite element exterior calculus allowing to choose any family without modification to the proofs.

Remark 3.2 (On Equations (3.4c) and (3.4d)). From the exterior calculus point of view ϕ and χ are harmonic forms of V^3, in practice they can be viewed as Lagrange multipliers. For the given example $\mathfrak{H}^3$ is trivial but if instead we consider $V^1 = H_0(\mathrm{curl}, \Omega)$, $V^2 = H_0(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$ then we must have $\mathfrak{H}^3 \approx \mathbb{R}$ (the space of constant functions). Equation (3.4c) ensures that the system is onto, although here the right-hand side is null so ϕ^n will always be zero. Equation (3.4d) ensures that p^n is orthogonal to the space of harmonic 3-forms (i.e. here that $\int_\Omega p^n = 0$).

Abstracting the linearization and time discretization scheme we simply consider two linear maps: l_3 and l_5 defined on $L^2(\Omega) \to L^2(\Omega)$ (the name of which follows the convention of Arnold & Li [45]). And we define the problem: Given $f_2, f_3 \in L^2(\Omega)$, find $(\omega, u, p, \phi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall(\tau, v, q, \chi) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$,

$$\begin{aligned} \langle \omega, \tau \rangle - \langle u, \nabla \times \tau \rangle &= 0, \\ \langle \nu \nabla \times \omega + l_3 \omega + l_5 u, v \rangle - \langle p, \nabla \cdot v \rangle &= \langle f_2, v \rangle, \\ \langle \nabla \cdot u, q \rangle + \langle \phi, q \rangle &= \langle f_3, q \rangle, \\ \langle p, \chi \rangle &= 0. \end{aligned} \tag{3.5}$$

We easily see that a suitable choice of l_3 and l_5 (namely $l_3 = (v \to \frac{1}{2} v \times u^{n-1})$ and $l_5 = (v \to \frac{1}{\delta t} v + \frac{1}{2} \omega^{n-1} \times v)$ allows us to recover (3.4) for $\theta = \frac{1}{2}$. We redefine the problem in the framework of exterior calculus in (3.26). Under mild assumptions on l_3, l_5 and Ω detailed in (3.23) we prove the well-posedness of the problem (3.26) (or equivalently (3.5)) and of its discrete counterpart (3.32). If we write (ω, u, p, ϕ) (resp. $(\omega_h, u_h, p_h, \phi_h)$) the solution of (3.26) (resp. (3.32)) then we derive an optimal a priori error estimate on the energy norm proportional to the approximation properties of the discrete spaces used. The result is stated in Corollary 3.28. When $f_3 = 0$ (which is the case for (3.4)) we show that the velocity is exactly divergence free even at the discrete level, i.e. that $\nabla \cdot u = \nabla \cdot u_h = 0$ holds pointwise. We also show that the scheme is pressure-robust (see Section 3.5.2). This allows deriving an error estimate for the vorticity and velocity independent of the pressure in Theorem 3.34.

The remaining of this paper is divided as follows: We define the notation used in the paper and discuss some applications of the scheme in Section 3.2. We show the well-posedness and derive error estimates for an intermediary problem akin to a Stokes problem in Section 3.3. Section 3.4 is dedicated to the analysis of problem (3.5). This is the most technical section. We derive some additional results in Section 3.5, and finally we present a variety of numerical simulations done with our scheme in Section 3.6 that validate our results and give some perspectives. The exterior calculus formalism is introduced in Section 3.2 and heavily used in Section 3.3 and 3.4. We assume that the reader has some familiarity with exterior calculus in those two sections (3.3 and 3.4). However, no prior knowledge of exterior calculus is expected in Section 3.5 and 3.6.

3.2 Setting

The exterior calculus framework allows getting a uniform vision on many objects. As such it may appear abstract and confusing at first. Therefore, we will start by giving some explicit examples of spaces to fix the ideas, and discuss how to deal with boundary conditions. Only then we will introduce the full notation and specifications of exterior calculus which will be used in the remaining part of the paper.

Throughout the paper we consider the problem in 3 dimensions. However, with usual modifications for the definition of ω, everything said and proved also works in 2 dimensions. This is an advantage of exterior calculus formalism.

3.2.1 Function spaces

Our scheme does not rely on a particular choice of discrete spaces, instead we make some assumptions (given in Section 3.2.3) on them and any spaces fulfilling these assumptions can be used. Adequate spaces are readily available on simplicial and cubic meshes (they are given e.g. in the periodic table of finite elements [36]). We illustrate here an example of sensible choices. Since the discrete spaces depend on the continuous ones (through boundary conditions) we begin by setting the continuous spaces for this example. Let $V^1 = H_0(\mathrm{curl}, \Omega)$, $V^2 = H_0(\mathrm{div}, \Omega)$, $V^3 = L^2(\Omega)$ and $\mathfrak{H}^3 = \mathbb{R} \subset L^2(\Omega)$. The discrete spaces depend on a polynomial degree $r \in \mathbb{N}$. Let $\mathbf{T}_h$ be a simplicial triangulation of Ω, we choose the following discretization:

- The curl space V_h^1 is built upon Nedelec's edge elements of the first kind of degree r (or $P_r^-\Lambda^1$ in the periodic table), $V_h^1 = \{\omega \in H_0(\mathrm{curl}, \Omega);\ \omega_{|T} \in P_r^-\Lambda^1(T), \forall T \in \mathbf{T}_h\}$.
- The velocity space V_h^2 is built upon Nedelec's face elements of the first kind of degree r (or $P_r^-\Lambda^2$), $V_h^2 = \{u \in H_0(\mathrm{div}, \Omega);\ u_{|T} \in P_r^-\Lambda^2(T), \forall T \in \mathbf{T}_h\}$.

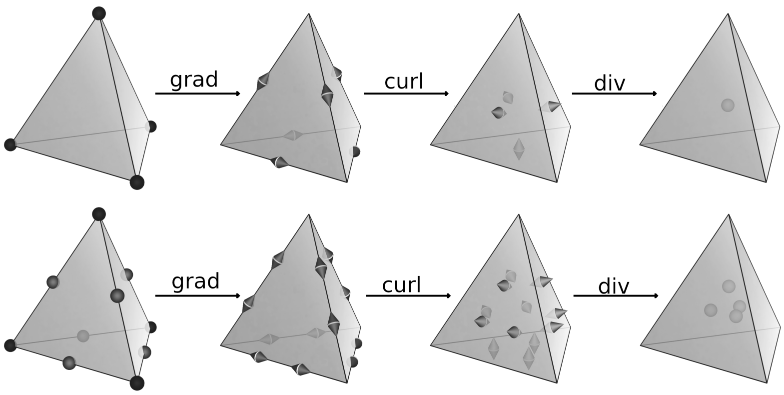


Figure 3.1: Degrees of freedom on reference elements for the polynomial degrees $r = 1$ and $r = 2$.

- The pressure space V_h^3 is built upon discontinuous Galerkin elements of degree $r-1$ (or $P_r^-\Lambda^3$), $V_h^3 = \{p \in L^2(\Omega);\ p_{|T} \in P_r^-\Lambda^3(T), \forall T \in \mathbf{T}_h\}$.
- The space of discrete harmonic forms $\mathfrak{H}_h^3 = \mathbb{R} \subset L^2(\Omega)$ is the L^2-orthogonal complement of $\mathrm{div}(V_h^2)$ in V_h^3.

We can replace the first kind with the second, at the expense of increasing the polynomial degrees. Nedelec elements of first and second kind are respectively the 3-dimensional equivalent of Raviart-Thomas and of Brezzi-Douglas-Marini elements. The space $\mathfrak{H}_h^3$ is just the natural way of fixing the pressure (which is defined up to an arbitrary constant when the normal component of the velocity is fixed on the boundary). Figure 3.1 shows the degrees of freedom of this choice of finite elements for $r = 1$ and $r = 2$. The first element shown on the left corresponds to the space V_h^0 that plays no role here.

Remark 3.3. In full generality $\mathfrak{H}_h^3$ is simply the space of the discrete harmonic 3-forms. If we were to take $V^1 = H(\mathrm{curl}, \Omega)$, $V^2 = H(\mathrm{div}, \Omega)$ and $V^3 = L^2(\Omega)$ instead, we would have $\mathfrak{H}^3 = \mathfrak{H}_h^3 = \{0\}$.

For this choice we can make explicit the approximation properties in terms of the size of the mesh h and of the polynomial degree r. For (ω, u, p, ϕ) (resp. $(\omega_h, u_h, p_h, \phi_h)$) the solution of (3.26) (resp. (3.32)) the estimate of Corollary 3.28 reads:

$$\|\omega - \omega_h\|_{H(\mathrm{curl})} + \|u - u_h\|_{H(\mathrm{div})} + \|p - p_h\|_{L^2} + \|\phi - \phi_h\|_{L^2} = \mathcal{O}(h^r).$$

3.2.2 Boundary conditions

We give a quick review of the boundary conditions readily available. Let n be the unit outward vector normal to the boundary and g and h arbitrary functions in a suitable space. Equalities below are always understood on $\partial\Omega$ and any of the following conditions can be used on some part of the boundary:

$$\omega \times n = g, \; u \cdot n = h, \tag{3.6}$$
$$\omega \times n = g, \; p = h, \tag{3.7}$$
$$u = g, \tag{3.8}$$
$$u \times n = g, \; p = h. \tag{3.9}$$

Condition (3.8) allows enforcing the no slip condition with $u \cdot n = 0$, $u \times n = 0$.

In order to implement them, we use essential Dirichlet boundary conditions for:

- $\omega \times n = g$ and take test functions in $H_0(\text{curl})$.
- $u \cdot n = g$ and take test functions in $H_0(\text{div})$.

And natural conditions for the other two:

- $u \times n = g$ by adding $-\int_{\partial\Omega}(g \times \tau) \cdot n ds$ to the left-hand side of equation (3.4a).
- $p = h$ by adding $\int_{\partial\Omega} h \, v \cdot n ds$ to the left-hand side of equation (3.4b).

Remark 3.4. Although they can all be imposed, only conditions (3.6) and (3.9) give a complex. The following proofs rely on the use of complexes, so we assume that one of these two conditions is chosen. When we do not use a complex we still have the well-posedness, however the rate of convergence can be impacted. The case of condition (3.8), which is used to impose the no slip condition, is studied by Arnold et al. [27] and Chen et al. [38]. They show that the rate of convergence on the vorticity (for the $H(\text{curl})$ norm) and on the pressure (for the L^2 norm) is slightly degraded. However, for a polynomial degree of at least 2 the rate of convergence of the velocity (for the $H(\text{div})$ norm) is not impacted.

3.2.3 Notation

Finally we state the full framework of exterior calculus which will be used. We make extensive use of the following notation introduced by D. Arnold [50]. They are explained in greater detail in the original reference.

- d is the exterior derivative, d^* its adjoint or codifferential,
- $W^0 \to W^1 \to W^2 \to W^3$ is the L^2 de Rham complex of a bounded domain of $\mathbb{R}^3$. We shall only use the last three spaces.

- $V^0 \to V^1 \to V^2 \to V^3$ is a dense subcomplex on which the exterior derivative is defined (and not just densely defined). Here we have $H^1(\Omega) \to H(\mathrm{curl}, \Omega) \to H(\mathrm{div}, \Omega) \to L^2(\Omega)$.
- V_k^* is the domain of the adjoint of (d, V^k).
- $\|\cdot\|$ is the L^2-norm for scalar or vector valued functions.
- $\|\cdot\|_V$ is the V norm, defined by $\|\cdot\|_V := \|\cdot\| + \|d\cdot\|$.
- Sometimes we take the norm $\|\cdot\|_{V\cap V^*}$ on $V\cap V^*$, it is given by $\|\cdot\|_V + \|d^*\cdot\|$.
- $\|(u,v)\|_{A\times B} := \|u\|_A + \|v\|_B$ is the norm on the product space $A\times B$.
- $V_h^0 \to V_h^1 \to V_h^2 \to V_h^3$ is a discrete subcomplex parametrized by h.
- P_h is the L^2-orthogonal projection on the discrete subcomplex, and in general P_A is the L^2-orthogonal projection on A.

We assume that our complexes have the compactness property, which means that the inclusion $V^k \cap V_k^* \subset W^k$ is compact for each k. We also assume that there exists a cochain projection $\pi_h^k : V^k \to V_h^k$, bounded for the W-norm, uniformly in h. These two properties hold for the De Rham complex on a bounded domain with Lipschitz boundary (see e.g. [24, Section 4.2] for the compactness, and [19, Section 5.4] or [22] for the bounded projection). Each space has the Hodge-decomposition $W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*$, where $\mathfrak{B}^k := d(V^{k-1})$, $\mathfrak{B}_k^* := d^*(V_{k+1}^*)$ and $\mathfrak{H}^k := \mathfrak{Z}^k \cap (\mathfrak{B}^k)^\perp$, $\mathfrak{Z}^k$ being the kernel of $d : V^k \to V^{k+1}$. This can also be written as $\mathfrak{Z}^k = \mathfrak{B}^k \oplus \mathfrak{H}^k$, and $W^k = \mathfrak{Z}^k \oplus \mathfrak{B}_k^*$.

In order to measure the approximation properties of the discrete subspaces we introduce the notation $E := E^k$, defined by:

$$\forall k, \forall \sigma \in V^k,\ E^k(\sigma) := \inf_{\tau\in V_h^k} \|\sigma - \tau\| \ . \tag{3.10}$$

The choice of spaces introduced in Section 3.2.1 does indeed verify all these properties with $E = \mathcal{O}(h^r)$ on a dense subset (see [24]). Each space has a Poincaré inequality (see [50, Theorem 4.6]). That is to say, for each k, $\exists c_p^k > 0$ such that $\forall z \in \mathfrak{Z}^{k\perp_{V^k}}$, $\|z\|_{V^k} \le c_p^k \|dz\|_{L^2}$. Moreover this also holds for the discrete subcomplex with constants bounded by $c_p^k\, \|\pi_h^k\|_{W^k}$ (see [50, Theorem 5.3]). We set $c_p := \max_{k,h} c_p^k\, \|\pi_h^k\|_{W^k}$, so that the Poincaré inequality holds for c_p regardless of the space.

For reference let $v_1 \in V^1$, $v_2 \in V^2$, $v_3 \in V^3$. The exterior derivative acts as $dv_1 := \mathrm{curl}(v_1)$, $dv_2 := \mathrm{div}(v_2)$ and $dv_3 := 0$. When it is obvious from the context we shall drop the exponent, hence $\pi_h v_1 := \pi_h^1 v_1$, $\pi_h v_2 := \pi_h^2 v_2$. When we apply an operator (such as π_h, d or d^*) to a product space, we mean to apply it to each component (i.e. $d(v_1, v_2, v_3) := (dv_1, dv_2, 0)$). When we add a suffix $_h$ such as d_h instead of d we refer to the discrete counterpart of the object. Since we use a discrete subcomplex, d_h is just the restriction of d to V_h. We will often add a numerical suffix such as z_2 for $z \in V^2 \times V^3$, this means we take the V^2 component of z.

Usually when dealing with a primal formulation we will use the variables (u, p), and they can indeed be seen as the velocity and pressure. However, when we deal with mixed formulations we will frequently write (u_1, u_2, u_3). Here 1,2 and 3 refer to 1-form, 2-form and 3-form and have nothing to do with components in a frame of the velocity field. Specifically we will have the identification $u_1 = \omega$, $u_2 = u$ and $u_3 = p$.

The symbol $A \lesssim B$ means that there exists a constant $C > 0$ independent of A and B (depending solely on few specified parameters) such that $A \leq CB$.

3.3 Linear steady problem

We first study a simpler problem, analogous to a Stokes problem, which is also closely related to the Hodge-Laplace problem (see [50]).

Definition 3.5. Let $D_0 := \{(u, p) \in (V^2 \cap V_2^*) \times V_3^* |\ d^*u \in V^1\}$, $f \in W^2 \times W^3$,

$$L_0 := \begin{bmatrix} \nu dd^* & -d^* \\ d & 0 \end{bmatrix} = \begin{bmatrix} \nu\nabla \times \nabla\times & \nabla \\ \nabla\cdot & 0 \end{bmatrix}. \tag{3.11}$$

The problem is to find $(u, p) \in P_{(\mathfrak{H}^2 \times \mathfrak{H}^3)^\perp} D_0$, such that

$$\forall (v, q) \in P_{(\mathfrak{H}^2 \times \mathfrak{H}^3)^\perp} D_0,\ \langle L_0(u, p), (v, q)\rangle = \langle f, (v, q)\rangle. \tag{3.12}$$

In vector calculus notation we want (u, p) such that

$$\nu\nabla \times (\nabla \times u) + \nabla p = f_2, \quad \nabla \cdot u = f_3.$$

3.3.1 Continuous well-posedness

Now we introduce the mixed formulation. Recall that u_2 (resp. u_3) defined below corresponds to u (resp. p) in (3.12). The mixed formulation is characterized by the introduction of the auxiliary variable u_1 corresponding to $\nabla \times u_2$ or d^*u_2. We define B_0 by:

$$\begin{aligned} &B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) := \\ &\quad \langle u_1, v_1\rangle - \langle u_2, dv_1\rangle + \nu\langle du_1, v_2\rangle - \langle u_3, dv_2\rangle + \langle \phi_2, v_2\rangle \\ &\quad + \langle du_2, v_3\rangle + \langle \phi_3, v_3\rangle + \langle u_2, \chi_2\rangle + \langle u_3, \chi_3\rangle. \end{aligned} \tag{3.13}$$

The problem reads: Given $(f_2, f_3) \in W^2 \times W^3$, find $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ such that $\forall (v_1, v_2, v_3, \chi_2, \chi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$,

$$B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) = \langle f_2, v_2 \rangle + \langle f_3, v_3 \rangle. \tag{3.14}$$

Remark 3.6. We removed d^* from the formulation to prepare for the discrete formulation (3.18). Indeed d^* and d_h^* are barely related. In fact d_h^* is a global operator which would greatly deteriorate the sparsity pattern of the system.

Lemma 3.7. *There is $\alpha > 0$, such that $\forall (u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$*

$$\sup_{(v_1,v_2,v_3,\chi_2,\chi_3)} \frac{B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3))}{\|(v_1, v_2, v_3, \chi_2, \chi_3)\|_V} \geq \alpha \|(u_1, u_2, u_3, \phi_2, \phi_3)\|_V.$$

Proof. For any $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$, let $\rho_1 \in \mathfrak{B}_1^* \cap V^1$ be such that $d\rho_1 = P_{\mathfrak{B}} u_2$, and $\rho_2 \in \mathfrak{B}_2^* \cap V^2$ be such that $d\rho_2 = P_{\mathfrak{B}} u_3 + du_2$. And take $v_1 = \nu u_1 - \frac{\nu}{c_p^2}\rho_1$, $v_2 = P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2$, $v_3 = du_2 - P_{\mathfrak{B}} u_3 + \phi_3$, $\chi_2 = P_{\mathfrak{H}} u_2$, $\chi_3 = P_{\mathfrak{H}} u_3$. The Poincaré inequality gives:

$$\|\rho_1\|_{V^1} \leq c_p \|P_{\mathfrak{B}} u_2\|, \quad \|\rho_2\|_{V^2} \leq c_p \|P_{\mathfrak{B}} u_3\| + c_p \|du_2\|. \tag{3.15}$$

And we easily see that, for hidden constants depending only on ν and c_p,

$$\|v_1\|_{V^1} + \|v_2\|_{V^2} + \|v_3\|_{V^3} + \|\chi_2\| + \|\chi_3\| \lesssim \|u_1\|_{V^1} + \|u_2\|_{V^2} + \|u_3\|_{V^3} + \|\phi_2\| + \|\phi_3\|.$$

Using the orthogonality of the Hodge decomposition we get:

$$\begin{aligned}
&B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) \\
&= \langle u_1, \nu u_1 - \frac{\nu}{c_p^2}\rho_1 \rangle - \langle u_2, d(\nu u_1 - \frac{\nu}{c_p^2}\rho_1) \rangle + \nu \langle du_1, P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2 \rangle \\
&\quad - \langle u_3, d(P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2) \rangle + \langle du_2, du_2 - P_{\mathfrak{B}} u_3 + \phi_3 \rangle + \langle u_2, P_{\mathfrak{H}} u_2 \rangle \\
&\quad + \langle u_3, P_{\mathfrak{H}} u_3 \rangle + \langle \phi_2, P_{\mathfrak{B}} u_2 + du_1 - \rho_2 + \phi_2 \rangle + \langle \phi_3, du_2 - P_{\mathfrak{B}} u_3 + \phi_3 \rangle \\
&= \nu \langle u_1, u_1 \rangle - \frac{\nu}{c_p^2} \langle u_1, \rho_1 \rangle - \nu \langle u_2, du_1 \rangle + \frac{\nu}{c_p^2} \langle u_2, P_{\mathfrak{B}} u_2 \rangle + \nu \langle du_1, u_2 \rangle \\
&\quad + \nu \langle du_1, du_1 \rangle + \langle u_3, P_{\mathfrak{B}} u_3 \rangle + \langle u_3, du_2 \rangle + \langle du_2, du_2 \rangle - \langle du_2, u_3 \rangle \\
&\quad + \langle u_2, P_{\mathfrak{H}} u_2 \rangle + \langle u_3, P_{\mathfrak{H}} u_3 \rangle + \langle \phi_2, \phi_2 \rangle + \langle \phi_3, \phi_3 \rangle \\
&= \nu \|u_1\|^2 - \frac{\nu}{c_p^2} \langle u_1, \rho_1 \rangle + \frac{\nu}{c_p^2} \|P_{\mathfrak{B}} u_2\|^2 + \nu \|du_1\|^2 + \|P_{\mathfrak{B}} u_3\|^2 + \|du_2\|^2 \\
&\quad + \|P_{\mathfrak{H}} u_2\|^2 + \|P_{\mathfrak{H}} u_3\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2.
\end{aligned} \tag{3.16}$$

Applying (3.15) and $\langle u_1, \rho_1 \rangle \leq \|u_1\| \|\rho_1\| \leq \frac{c_p^2}{2} \|u_1\|^2 + \frac{1}{2c_p^2} \|\rho_1\|^2$ to (3.16) yields:

$$\begin{aligned}
B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) &\geq \frac{\nu}{2} \|u_1\|^2 + \frac{\nu}{2c_p^2} \|P_{\mathfrak{B}} u_2\|^2 \\
&+ \nu \|du_1\|^2 + \|u_3\|^2 + \|du_2\|^2 + \|P_{\mathfrak{H}} u_2\|^2 + \|P_{\mathfrak{H}} u_3\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2.
\end{aligned}$$

Finally, $du_3 = 0$, $d\phi_2 = d\phi_3 = 0$ and $\|P_{\mathfrak{B}^*}u_2\| \leq c_p\|dP_{\mathfrak{B}^*}u_2\| = c_p\|du_2\|$, hence

$$B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) \gtrsim \|(u_1, u_2, u_3, \phi_2, \phi_3)\|_V^2,$$

where the hidden constant depends only on c_p and ν. □

Lemma 3.8. *Given any* $(v_1, v_2, v_3, \chi_2, \chi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ *non-zero, there is* $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ *such that*

$$B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_1, v_2, v_3, \chi_2, \chi_3)) > 0\ .$$

Proof. If $P_{\mathfrak{B}}v_2 \neq 0$ take $u_1 \in \mathfrak{B}_1^*$ such that $du_1 = P_{\mathfrak{B}}v_2$, $\phi_2 = 0$, $u_3 = \phi_3 = 0$. Then if $\langle u_1, v_1\rangle = 0$ take $u_2 = 0$ else take $u_2 = dv_1\frac{\langle u_1, v_1\rangle}{\langle dv_1, dv_1\rangle}$ ($dv_1 \neq 0$ since $P_{\mathfrak{B}^*}v_1 \neq 0$ as $\langle u_1, v_1\rangle \neq 0$). If $P_{\mathfrak{B}}v_2 = 0$, simply take $u_1 = v_1$, u_2 such that $du_2 = P_{\mathfrak{B}}v_3, P_{\mathfrak{B}}u_2 = -dv_1, P_{\mathfrak{H}}u_2 = \chi_2$ (this is possible by the Hodge decomposition), u_3 such that $P_{\mathfrak{B}}u_3 = -dv_2, P_{\mathfrak{H}}u_3 = \chi_3$, $\phi_2 = P_{\mathfrak{H}}v_2$ and $\phi_3 = P_{\mathfrak{H}}v_3$. □

Lemma 3.7 and Lemma 3.8 together with the Babuška–Lax–Milgram theorem give the well-posedness of (3.14). Moreover, we have for c depending only on c_p and ν:

$$\|u_1\|_{V^1} + \|u_2\|_{V^2} + \|u_3\|_{V^3} + \|\phi_2\|_{V^2} + \|\phi_3\|_{V^3} \leq c\,\|f\|. \tag{3.17}$$

3.3.2 Discrete problem and error estimate

The discrete problem reads:
Given $(f_2, f_3) \in W^2 \times W^3$, find $(u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ such that $\forall(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$,

$$B_0((u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h}), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) = \langle f_2, v_{2h}\rangle + \langle f_3, v_{3h}\rangle. \tag{3.18}$$

Lemma 3.9. *Problem* (3.18) *is well-posed.*

Proof. Since we have a discrete Poincaré inequality (see [50]) and use a subcomplex we can apply exactly the same proof as in the continuous case. □

Next we derive a basic error estimate for a global norm. Improved error estimates for each component are given by Theorem 3.38. First we define:

$$\mu := \max_{k\in\{2,3\}} \sup_{r\in\mathfrak{H}^k, \|r\|=1} \|(I - \pi_h^k)r\|. \tag{3.19}$$

Theorem 3.10. *Given* $(f_2, f_3) \in W^2 \times W^3$*, let* $(u_1, u_2, u_3, \phi_2, \phi_3)$ *be the solution of the continuous problem* (3.14) *and* $(u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h})$ *the solution of the discrete problem* (3.18)*, then for* E *given by* (3.10) *it holds:*

$$\begin{aligned}
&\|(u_1 - u_{1h}, u_2 - u_{2h}, u_3 - u_{3h})\|_V + \|(\phi_2 - \phi_{2h}, \phi_3 - \phi_{3h})\| \\
&\lesssim \inf_{v_1\in V_h^1} \|u_1 - v_1\|_{V^1} + \inf_{v_2\in V_h^2} \|u_2 - v_2\|_{V^2} + E(u_3) + E(\phi_2) + E(\phi_3) \\
&\quad + \mu(E(P_{\mathfrak{B}}u_2) + E(P_{\mathfrak{B}}u_3)).
\end{aligned}$$

Proof. For all $(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ we have:

$$\begin{aligned} B_0((u_1, u_2, u_3, \phi_2, \phi_3), (v_{1h}, &v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &= \langle f_2, v_{2h}\rangle + \langle f_3, v_{3h}\rangle + \langle u_2, \chi_{2h}\rangle + \langle u_3, \chi_{3h}\rangle \ . \end{aligned}$$

Let $(v_1, v_2, v_3, \chi_2, \chi_3)$ be the V-orthogonal projection of $(u_1, u_2, u_3, \phi_2, \phi_3)$ into their respective discrete spaces. By the continuity of B_0 it holds, $\forall(v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$,

$$\begin{aligned} B_0(&(u_{1h} - v_1, u_{2h} - v_2, u_{3h} - v_3, \phi_{2h} - \chi_2, \phi_{3h} - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &= B_0((u_1 - v_1, u_2 - v_2, u_3 - v_3, \phi_2 - \chi_2, \phi_3 - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &\quad - \langle u_2, \chi_{2h}\rangle - \langle u_3, \chi_{3h}\rangle \\ &= B_0((u_1 - v_1, u_2 - v_2, u_3 - v_3, \phi_2 - \chi_2, \phi_3 - \chi_3), (v_{1h}, v_{2h}, v_{3h}, \chi_{2h}, \chi_{3h})) \\ &\quad - \langle P_{\mathfrak{H}_h} u_2, \chi_{2h}\rangle - \langle P_{\mathfrak{H}_h} u_3, \chi_{3h}\rangle \\ &\lesssim (\|u_1 - v_1\|_{V^1} + \|u_2 - v_2\|_{V^2} + \|u_3 - v_3\| + \|\phi_2 - \chi_2\| + \|\phi_3 - \chi_3\| \\ &\quad + \|P_{\mathfrak{H}_h} u_2\|_{V^2} + \|P_{\mathfrak{H}_h} u_3\|_{V^3})(\|v_{1h}\|_{V^1} + \|v_{2h}\|_{V^2} + \|v_{3h}\| + \|\chi_{2h}\| + \|\chi_{3h}\|), \end{aligned}$$

where the hidden constant depends only on ν.

From the discrete inf-sup condition we have

$$\begin{aligned} (\|u_{1h} - v_1\|_{V^1} + \|u_{2h} - v_2\|_{V^2} + \|u_{3h} - v_3\| + \|\phi_{2h} - \chi_2\| + \|\phi_{3h} - \chi_3\|) \\ \lesssim (\|u_1 - v_1\|_{V^1} + \|u_2 - v_2\|_{V^2} + \|u_3 - v_3\| + \|\phi_2 - \chi_2\| \\ + \|\phi_3 - \chi_3\| + \|P_{\mathfrak{H}_h} u_2\|_{V^2} + \|P_{\mathfrak{H}_h} u_3\|), \end{aligned}$$

where the hidden constant depends only on ν and on the discrete constant of Poincaré. The theorem is inferred from [50, Theorem 5.2] and [24, Equation (33)] which state:

$$\|P_{\mathfrak{H}_h} u_i\|_V \lesssim \mu\, E(P_{\mathfrak{B}} u_i), \quad \|\phi_i - \chi_i\| \lesssim E(\phi_i), \quad \forall i \in \{2, 3\}.$$

□

Remark 3.11. $E(\phi_i)$ is understood as viewing ϕ_i as an element of $V^i \supset \mathfrak{H}^i$,

$$E(\phi_i) := \inf_{q \in V_h^i} \|\phi_i - q\|.$$

3.4 Linearized problem

We construct our scheme by adding some lower order terms to Problem (3.14). We recall the correspondence with the names used for variables in the introduction:

$(u_1, u_2, u_3, u_p) \equiv (\omega, u, p, \phi)$ in (3.5). To keep the notation bearable we shall also write $u_{\mathfrak{B}} := P_{\mathfrak{B}}u$, $u_{\mathfrak{H}} := P_{\mathfrak{H}}u$ and so on.

We consider two linear maps $l_3 : W^1 \to W^2$ and $l_5 : W^2 \to W^2$ (we chose these names to match those used by Arnold & Li [45]). Define $D := \{(u,p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) |\ d^*u \in V^1\}$, $W = W^2 \times (W^3 \cap \mathfrak{H}^{3\perp})$ and

$$L := \begin{bmatrix} (\nu d + l_3)d^* + l_5 & -d^* \\ d & 0 \end{bmatrix}. \tag{3.20}$$

We consider the primal problem: Given $f \in W$, find $(u,p) \in D$ such that

$$L(u,p) = f. \tag{3.21}$$

We also define the dual operator L' on $D' := \{(u,p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) |\ (\nu d^* + l_3^*)u \in V^1\}$ by:

$$L' := \begin{bmatrix} d(\nu d^* + l_3^*) + l_5^* & d^* \\ -d & 0 \end{bmatrix}. \tag{3.22}$$

As an intermediary step, we wish to extend L to a larger domain and introduce:

$$\overline{L_\lambda} : (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))',$$
$$\overline{L_\lambda}(u,p)(v,q) := \langle \nu d^*u, d^*v\rangle + \langle l_3 d^*u + l_5 u - \lambda d^*p, v\rangle + \langle \lambda du, q\rangle,$$

$$\overline{L'_\lambda} : (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))',$$
$$\overline{L'_\lambda}(u,p)(v,q) := \langle \nu d^*u + l_3^* u, d^*v\rangle + \langle l_5^* u + \lambda d^*p, v\rangle + \langle -\lambda du, q\rangle.$$

Here λ is a positive parameter introduced to simplify the proof of Theorem 3.12. In Remark 3.15 we shall see that $\overline{L_1}$ and $\overline{L'_1}$ are almost always isomorphisms. We define the solution operator $K := (\overline{L'_1})^{-1}$, and we assume that

$$d^*(\overline{L_1}^{-1})_2(W) \subset V^1,\ (\nu d^* + l_3^*)(K)_2(W) \subset V^1, \tag{3.23}$$

where $(\overline{L_1}^{-1})_2$ and $(K)_2$ are the projections on the first component of the product space taken after the operators. Moreover, we assume that $\|dd^*(K)_2\|_{W\to W^2}$ and $\|dl_3^*(K)_2\|_{W\to W^2}$ are bounded. We show in Section 3.5.1 that these assumptions are satisfied when l_3 and l_5 are those used in our scheme.

The proof follows the same outline as [45]. First we prove that the continuous primal formulation gives an isomorphism, then we prove that the continuous mixed formulation is well-posed. Lastly we prove the well-posedness of the discrete mixed formulation and give an estimation of the error in energy norm.

3.4.1 Continuous primal formulation

Theorem 3.12. $\overline{L_\lambda} + \gamma\langle\cdot,\cdot\rangle$ *is a bounded isomorphism for all* $\gamma \in \mathbb{R}$ *except for a countable subset.*

Proof. Let $c = max(\|l_3\|, \|l_5\|)$, for $(u,p) \in (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp})$ take $v_{\mathfrak{B}} = u_{\mathfrak{B}}$, $v_{\mathfrak{H}} = u_{\mathfrak{H}}$, $v_{\mathfrak{B}^*} = -d^*p$, $q = du$. We have:

$$\begin{aligned}(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &= \nu\langle d^*u, d^*u\rangle + \lambda\langle d^*p, d^*p\rangle + \lambda\langle du, du\rangle \\ &\quad + \langle l_3 d^*u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle \\ &\quad + \gamma\langle u, u_{\mathfrak{B}} + u_{\mathfrak{H}}\rangle - \gamma\langle u, d^*p\rangle + \gamma\langle p, du\rangle \\ &= \nu\|d^*u\|^2 + \lambda\|d^*p\|^2 + \lambda\|du\|^2 + \gamma\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 \\ &\quad + \langle l_3 d^*u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle + \langle l_5 u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle.\end{aligned} \tag{3.24}$$

We bound the last line from (3.24) with the Cauchy-Schwarz inequality:

$$\begin{aligned}&|\langle l_3 d^*u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle| + |\langle l_5 u, u_{\mathfrak{B}} - d^*p + u_{\mathfrak{H}}\rangle| \\ &\quad \leq \frac{c^2}{2\nu}(\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \|d^*p\|^2) + \frac{\nu}{2}\|d^*u\|^2 + \frac{c^2}{2}(\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \|d^*p\|^2) + \frac{1}{2}\|u\|^2.\end{aligned}$$

Since $\|u_{\mathfrak{B}^*}\|^2 \leq c_p^2\|du\|^2$ and $\|u\|^2 = \|u_{\mathfrak{B}^*}\|^2 + \|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2$ we have:

$$\begin{aligned}(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &\geq (\gamma - \frac{c^2}{2\nu} - \frac{c^2}{2} - \frac{1}{2})\|u_{\mathfrak{B}} + u_{\mathfrak{H}}\|^2 + \frac{\nu}{2}\|d^*u\|^2 \\ &\quad + (\lambda - \frac{c^2}{2\nu} - \frac{c^2}{2})\|d^*p\|^2 + (\lambda - \frac{c_p^2}{2})\|du\|^2.\end{aligned}$$

We use the Poincaré inequality to bound $\|u_{\mathfrak{B}^*}\|$ by $\|du\|$ and $\|p_{\mathfrak{B}}\|$ by $\|d^*p\|$ (on the dual complex). Since $\|p_{\mathfrak{B}}\| = \|p\|$ (as $p \in V^3 \cap \mathfrak{H}^{3\perp}$) we have for λ and γ large enough (solely depending on c, ν and c_p):

$$\begin{aligned}(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &\gtrsim \|u\|^2 + \|du\|^2 + \|d^*u\|^2 + \|p\|^2 + \|d^*p\|^2 \\ &\gtrsim \|(u,p)\|^2_{V^2 \cap V_2^* \times V_3^*}.\end{aligned}$$

Clearly $\|(v,q)\|_{V_2^* \times W^3} \leq \|(u,p)\|_{V^2 \cap V_2^* \times V_3^*}$ as $\|d^*v\| = \|d^*u_{\mathfrak{B}}\| \leq \|d^*u\|$ and $(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})$ is continuous as a bilinear form from $(V^2 \cap V_2^* \times V_3^*) \times (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))$.

The only thing left to show in order to use the Babuška–Lax–Milgram theorem is the second condition. For any $(v,q) \neq 0$ we must find (u,p) such that $(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) > 0$. We can take λ and γ such that $\lambda^2 = \gamma\nu$. We consider two cases:

When $v = 0$ take u solution of the Hodge-Dirac problem (see [46]): $du = q$, $d^*u = 0$. We readily check that $(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,0)(0,q) = \lambda\langle du, q\rangle = \lambda\langle q, q\rangle > 0$.

When $v \neq 0$, we can find $u \in \{w \in V^2 \cap V_2^* | d^*w \in V^1, dw \in V_3^*\}$ such that $((\nu d + l_3)d^* + l_5 + \gamma + \nu d^*d)u = v$ (see [45]). Take $p = -\nu/\lambda du$, then since $\lambda = \gamma\nu/\lambda$ it holds:

$$\begin{aligned}(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})(u,p)(v,q) &= \langle \nu dd^*u, v\rangle + \langle (l_3 d^* + l_5 + \gamma)u, v\rangle - \lambda\langle d^*p, v\rangle \\ &\quad + \lambda\langle du, q\rangle + \gamma\langle p, q\rangle \\ &= \langle ((\nu d + l_3)d^* + l_5 + \gamma + \nu d^*d)u, v\rangle \\ &\quad + \lambda\langle du, q\rangle - \frac{\gamma\nu}{\lambda}\langle du, q\rangle \\ &= \langle v, v\rangle .\end{aligned}$$

Thus $(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})$ is a bounded isomorphism from $(V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}^{3\perp})$ to $(V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))'$. Since $I : (V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}_3^{\perp}) \to (V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))'$ is compact by the compactness property, $I(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})^{-1}$ is also compact. Since the spectrum of a compact operator is at most countable, $Id + \eta I(\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda})^{-1}$ has a bounded inverse for all $\eta \in \mathbb{R}$ except for a countable subset. Therefore, by composing to the right with $\gamma\langle\cdot,\cdot\rangle + \overline{L_\lambda}$ we get that $\overline{L_\lambda} + (\gamma+\eta)I$ has almost always a bounded inverse. □

Hence, up to an arbitrarily small perturbation, $\overline{L_\lambda}$ is a bounded isomorphism from $(V^2 \cap V_2^*) \times (V_3^* \cap \mathfrak{H}_3^{\perp})$ to $(V_2^* \times (W^3 \cap \mathfrak{H}^{3\perp}))'$.

Remark 3.13. We could have left $\mathfrak{H}^3$ in the domain and the proof above would still work. However, in this case, $\overline{L_\lambda}$ would never have been an isomorphism since its image cannot reach $\mathfrak{H}^3$.

We have the same result for the dual problem.

Lemma 3.14. *For all $\gamma \in \mathbb{R}$ except for a countable subset, $\overline{L'_\lambda} + \gamma\langle\cdot,\cdot\rangle$ is a bounded isomorphism.*

Proof. The same proof as the one of Theorem 3.12 works. The only differences will be a sign in the chosen (v, q), l_5 and l_3 replaced with l_5^* and l_3^* and $(l_3 d^*u, v)$ changed to (l_3^*u, d^*v). This does not add any difficulty in the proof. □

Remark 3.15. The proof of Theorem 3.12 requires taking λ to be sufficiently large, however for any $f_2 \in (V_2^*)'$, $f_3 \in (W^3 \cap \mathfrak{H}^{3\perp})'$, $\lambda_0 > 0$, $\lambda_1 > 0$ we have the following equivalence:

$$\overline{L_{\lambda_0}}(u,p) = (f_2, f_3) \Leftrightarrow \overline{L_{\lambda_1}}(u, \frac{\lambda_0}{\lambda_1}p) = (f_2, \frac{\lambda_1}{\lambda_0}f_3) .$$

Therefore, if $\overline{L_{\lambda_0}}$ is a bounded isomorphism for a given λ_0, it easily follows that $\overline{L_{\lambda_1}}$ is a bounded isomorphism for any $\lambda_1 > 0$, in particular for $\lambda_1 = 1$. The same argument works for $\overline{L'_\lambda}$.

From here onward, we assume that $\overline{L_1}$ and $\overline{L'_1}$ are bounded isomorphisms.

3.4.2 Well-posedness of the continuous mixed formulation

As we did in the unperturbed case, we introduce an auxiliary variable in the problem. In the following, $\mathbf{u}$ is a shortcut for (u_1, u_2, u_3, u_p). We define B by:

$$\begin{aligned} B(\mathbf{u}, \mathbf{v}) := \langle u_1, v_1\rangle - \langle u_2, dv_1\rangle + \nu\langle du_1, v_2\rangle - \langle u_3, dv_2\rangle + \langle du_2, v_3\rangle \\ + \langle l_3 u_1, v_2\rangle + \langle l_5 u_2, v_2\rangle + \langle u_p, v_3\rangle + \langle u_3, v_p\rangle. \end{aligned} \tag{3.25}$$

The mixed formulation is: Given $(f_2, f_3) \in W$, find $\mathbf{u} := (u_1, u_2, u_3, u_p) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$ such that $\forall \mathbf{v} := (v_1, v_2, v_3, v_p) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$,

$$B(\mathbf{u}, \mathbf{v}) = \langle f_2, v_2\rangle + \langle f_3, v_3\rangle. \tag{3.26}$$

For (u, p) solution of (3.21), it immediately appears that $(d^*u, u, p, 0)$ solves (3.26). Now for (u_1, u_2, u_3, u_p) solution of (3.26), the first line implies that $u_2 \in V_2^*$, $d^*u_2 = u_1$ and therefore $d^*u_2 \in V^1$. The second line implies that $u_3 \in V_3^*$. And the last implies that $u_3 \perp \mathfrak{H}^3$. Therefore $(u_2, u_3) \in D$, and it obviously solves (3.21).

Theorem 3.16. *Under the regularity assumption* (3.23), *Problem* (3.26) *is well-posed.*

The proof follows from the Babuška–Lax–Milgram theorem along with Lemma 3.19 and inf-sup condition (3.31). In the following, hidden constants only depend on $\|l_3\|$, $\|l_5\|$, $\|K\|$, $\|d(\nu d^* + l_3^*)(K)_2\|$, ν and on constants of Poincaré. We shall write $V = V^1 \times V^2 \times V^3 \times \mathfrak{H}^3$.

Lemma 3.17. *For all* $(0, u_2, u_3, 0) = \mathbf{u} \in V$, *there exists* $\mathbf{z} \in V$ *such that* $\|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V$ *and* $\forall \omega \in V, B(\omega, \mathbf{z}) = \langle \omega, \mathbf{u}\rangle$.

Proof. Take $(z_2, z_3) = K(u_2, P_{\mathfrak{H}^\perp} u_3)$ and $\xi = -(\nu d^* + l_3^*)z_2$. Then $\xi \in V^1$ by assumption (3.23), and by the definition of $K := (\overline{L_1'})^{-1}$ we have $\forall \omega \in V$:

$$\begin{aligned} -\langle dz_2, \omega_3\rangle &= \langle P_{\mathfrak{H}^\perp} u_3, \omega_3\rangle, \\ \langle d(\nu d^* + l_3^*)z_2 + l_5^* z_2 + d^* z_3, \omega_2\rangle &= \langle u_2, \omega_2\rangle. \end{aligned} \tag{3.27}$$

We set $\mathbf{z} := (\xi, z_2, z_3, u_{3\mathfrak{H}})$. Applying (3.27) we have:

$$\begin{aligned} B(\omega, \mathbf{z}) &= \langle \omega_1, \xi\rangle - \langle \omega_2, d\xi\rangle + \langle (\nu d + l_3)\omega_1, z_2\rangle - \langle \omega_3, dz_2\rangle + \langle l_5\omega_2, z_2\rangle \\ &\quad + \langle d\omega_2, z_3\rangle + \langle \omega_3, u_{3\mathfrak{H}}\rangle + \langle \omega_p, z_3\rangle \\ &= \langle \omega_1, \xi\rangle - \langle \omega_2, d\xi\rangle + \langle \omega_1, (\nu d^* + l_3^*)z_2\rangle + \langle \omega_3, P_{\mathfrak{H}^\perp} u_3\rangle + \langle \omega_2, l_5^* z_2\rangle \\ &\quad + \langle \omega_2, d^* z_3\rangle + \langle \omega_3, u_{3\mathfrak{H}}\rangle \\ &= \langle \omega_1, \xi\rangle - \langle \omega_2, d\xi\rangle + \langle \omega_1, (\nu d^* + l_3^*)z_2\rangle - \langle \omega_2, d(\nu d^* + l_3^*)z_2\rangle \\ &\quad + \langle \omega_2, u_2\rangle + \langle \omega_3, P_{\mathfrak{H}^\perp} u_3\rangle + \langle \omega_3, u_{3\mathfrak{H}}\rangle \\ &= -\langle \omega_1, (\nu d^* + l_3^*)z_2\rangle + \langle \omega_2, d(\nu d^* + l_3^*)z_2\rangle \\ &\quad + \langle \omega_1, (\nu d^* + l_3^*)z_2\rangle - \langle \omega_2, d(\nu d^* + l_3^*)z_2\rangle + \langle \omega_3, u_3\rangle + \langle \omega_2, u_2\rangle \\ &= \langle \omega_2, u_2\rangle + \langle \omega_3, u_3\rangle. \end{aligned}$$

Moreover, since $\overline{L_1'}$ is a bounded isomorphism, so is K thus

$$\begin{aligned}\|\mathbf{z}\|_V &\le \|\nu d^* z_2\| + \|l_3^* z_2\| + \|z_2\|_{V^2} + \|z_3\| + \|d\xi\| + \|u_{3\mathfrak{H}}\| \\ &\lesssim \|z_2\|_{V^2\cap V_2^*} + \|z_3\|_{V_3^*} + \|d\xi\| + \|u_{3\mathfrak{H}}\| \\ &\lesssim (\|K\| + \|d(\nu d^* + l_3^*)K\| + 1)\|\mathbf{u}\| \lesssim \|\mathbf{u}\|_V.\end{aligned}$$

□

Lemma 3.18. *For all* $\mathbf{u} \in V$*, there exists* $\mathbf{z} \in V$ *such that* $\|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V$ *and* $B(\mathbf{u}, \mathbf{z}) \gtrsim \|du_1\|^2 + \|du_2\|^2 + \|u_1\|^2 + \|u_p\|^2 - \|u_2\|^2$.

Proof. Let $c = \max(\|l_3\|, \|l_5\|)$. We begin with some preliminary computations:

$$B(\mathbf{u}, (0, 0, u_p, 0)) = \langle u_p, u_p\rangle, \tag{3.28}$$

$$\begin{aligned}B(\mathbf{u}, (0, du_1, 0, 0)) &= \nu\langle du_1, du_1\rangle + \langle l_3 u_1, du_1\rangle + \langle l_5 u_2, du_1\rangle \\ &\ge \frac{1}{2}\nu\|du_1\|^2 - \frac{c^2}{\nu}(\|u_1\|^2 + \|u_2\|^2),\end{aligned} \tag{3.29}$$

$$\begin{aligned}B(\mathbf{u}, (\nu u_1, u_{2\mathfrak{B}}, du_2, 0)) &= \nu\langle u_1, u_1\rangle - \nu\langle u_2, du_1\rangle + \nu\langle du_1, u_{2\mathfrak{B}}\rangle \\ &\quad - \langle u_3, du_{2\mathfrak{B}}\rangle + \langle du_2, du_2\rangle + \langle l_3 u_1, u_{2\mathfrak{B}}\rangle \\ &\quad + \langle l_5 u_2, u_{2\mathfrak{B}}\rangle + \langle u_p, du_2\rangle + \langle u_3, 0\rangle \\ &\ge \frac{\nu}{2}\|u_1\|^2 + \|du_2\|^2 - \left(\frac{c^2}{2\nu} + c\right)\|u_2\|^2.\end{aligned} \tag{3.30}$$

Clearly it is possible to construct a suitable $\mathbf{z}$ from a linear combination of (3.28), (3.29) and (3.30). Bounds on norms are easily checked, for example:

$$\|(0, du_1, 0, 0)\|_V = \|du_1\| + \|ddu_1\| = \|du_1\| \lesssim \|\mathbf{u}\|_V.$$

□

Combining the two preceding lemmas gives:

$$\forall \mathbf{u} \in V, \sup_{\|\mathbf{v}\|_V = 1} |B(\mathbf{u}, \mathbf{v})| \gtrsim \|\mathbf{u}\|_V \ . \tag{3.31}$$

Lemma 3.19. *For any* $\mathbf{v} \in V$*, there is* $\mathbf{u} \in V$ *such that* $B(\mathbf{u}, \mathbf{v}) > 0$.

Proof. Given $\mathbf{v} \neq 0 \in V$, if $v_2 = 0$, $v_3 = 0$ and $v_p = 0$ take $\mathbf{u} = (v_1, 0, 0, 0)$, then

$$B(\mathbf{u}, \mathbf{v}) = \langle v_1, v_1\rangle > 0.$$

Else take $(u_2, u_3) = \overline{L_1}^{-1}(v_2, P_{\mathfrak{H}^\perp} v_3) + (0, v_p)$, $u_p = P_{\mathfrak{H}} v_3$ and $u_1 = d^* u_2$ ($u_1 \in V^1$ by assumption (3.23)) then

$$B(\mathbf{u}, \mathbf{v}) = \langle v_2, v_2\rangle + \langle v_3, v_3\rangle + \langle q, q\rangle > 0.$$

□

3.4.3 Discrete well-posedness

We introduce the notation $V_h = V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^3$. The discrete variational problem is the same as the continuous one, replacing V by V_h. Hence we shall still use the notation B, this time as a function from $V_h \times V_h$ to $\mathbb{R}$. So that the discrete problem is: Given $(f_2, f_3) \in W$, find $\mathbf{u}_h := (u_{1h}, u_{2h}, u_{3h}, u_{ph}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^3$ such that $\forall \mathbf{v}_h := (v_{1h}, v_{2h}, v_{3h}, v_{ph}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^3$,

$$B(\mathbf{u}_h, \mathbf{v}_h) = \langle f_2, v_{2h} \rangle + \langle f_3, v_{3h} \rangle. \tag{3.32}$$

Considering the dual of problem (3.12) with $\nu = 1$, we have: $D_0 := \{(u,p) \in (V^2 \cap V_2^*) \times V_*^3 |\ d^*u \in V^1\}$ and $L_0'(u,p) := (dd^*u + d^*p, -du)$. Let K_0 be the solution operator of the dual problem. We have $K_0 = (L_0')^{-1}$ when L_0' is viewed as an isomorphism from $P_{\mathfrak{H}^\perp} D_0$ to $P_{\mathfrak{H}^\perp}(W^2 \times W^3)$, and K_0 is extended by 0 on $\mathfrak{H}$. Explicitly we have the decomposition: $\forall (f_2, f_3) \in W^2 \times W^3$,

$$(f_2, f_3) = (dd^*(K_0)_2(f_2, f_3) + d^*(K_0)_3(f_2, f_3), -d(K_0)_2(f_2, f_3)) + (P_{\mathfrak{H}} f_2, P_{\mathfrak{H}} f_3)$$

and a similar expression for their discrete counterparts. Therefore,

$$\forall (z_2, z_3) \in D_0,\ (P_{\mathfrak{H}^\perp} z_2, P_{\mathfrak{H}^\perp} z_3) = L_0' K_0(z_2, z_3) = K_0 L_0'(z_2, z_3). \tag{3.33}$$

As mentioned before this problem is closely related to the one studied by Arnold & Li [45]. Since the mixed variable part is almost unchanged we shall use the generalized canonical projection Π_h (see [45]) and we state its properties below.

Lemma 3.20. *Under the condition of [45, Theorem 5.1]:*

- Π_h *is a projection uniformly bounded in the V-norm.*
- $d\Pi_h = P_{\mathfrak{B}_h} d$.
- $\forall w \in V^k, \|(I - \Pi_h)w\| \lesssim \|(I - \pi_h)w\| + \eta_0' \|dw\|$.
- $\forall w, v \in V^k, |\langle (I - \Pi_h)w, v \rangle| \lesssim (\|(I - \pi_h)w\| + \eta_0' \|dw\|)(\|(I - \pi_h)v\| + \eta_0' \|dv\|) + \alpha_0' \|dw\| \|dv\|$.

Here $\eta_0', \alpha_0' \to 0$ when $h \to 0$. They are given, along the proof in the reference [45].

Definition 3.21. We shall use the following notation in this section:

$$\delta_0 := \|(I - \pi_h)K_0\|,\ \mu_0 := \|(I - \pi_h)P_{\mathfrak{H}}\|,$$

$$\eta_0 := \max\{\|(I - \pi_h)dK_0\|, \|(I - \pi_h)d^*(K_0)_2\|\},$$

$$\alpha_0 := \eta_0(1 + \eta_0) + \mu_0 + \delta_0 + \mu_0\delta_0 + \eta_0',$$

$$\eta := \max\{\delta_0, \mu_0, \eta_0, \|(I - \pi_h)l_3^*(K)_2\|, \|(I - \pi_h)dl_3^*(K)_2\|\}.$$

Lemma 3.22. *The following bounds hold:*

$$\|K_0 - K_{0h}P_h\| \lesssim \alpha_0,$$

$$\|dK_0 - dK_{0h}P_h\| + \|d^*(K_0)_2 - d_h^*(K_{0h})_2 P_h\| \lesssim \eta_0.$$

Proof. Let $(f_2, f_3) \in (W^2 \times W^3)$. The idea is to apply the error estimate of Theorem 3.10 for $(u_2, u_3) = K_0(f_2, f_3)$, $(\phi_2, \phi_3) = P_{\mathfrak{H}}(f_2, f_3)$, $u_1 = d^*u_2$, $(u_{2h}, u_{3h}) = K_{0h}P_h(f_2, f_3)$, $(\phi_{2h}, \phi_{3h}) = P_{\mathfrak{H}_h}P_h(f_2, f_3)$, $u_{1h} = d_h^* u_{2h}$. Unfortunately we cannot conclude with the crude estimate of Theorem 3.10 because of the error on u_1. We need improved estimates that give

$$\begin{aligned}\|u_2 - u_{2h}\| + \|u_3 - u_{3h}\| &\lesssim (1+\mu_0)E(u_2) + E(u_3) + \eta_0 E(u_1)\\ &\qquad (\eta_0^2 + \delta_0 + \eta_0')E(du_1) + \eta_0' E(\phi_2) + E(du_2),\\ \|du_2 - du_{2h}\| + \|u_1 - u_{1h}\| &\lesssim E(du_2) + E(u_1) + \eta_0 E(du_1).\end{aligned}$$

We conclude since $E(u_2) + E(u_3) \leq \delta_0 \|(f_2, f_3)\|$, $E(u_1) + E(du_2) \leq \eta_0 \|(f_2, f_3)\|$ and $E(du_1) + E(\phi_2) \leq \|(f_2, f_3)\|$, the last coming from $du_1 = P_{\mathfrak{B}} f_2$. These proofs are technical and mostly follow those of [24, Theorem 3.11], see Appendix 3.A. □

Lemma 3.23. *For $f \in \mathfrak{H}^\perp$ we have $\|P_{\mathfrak{H}_h} f\| \lesssim \mu_0 \|f\|$.*

Proof. We recall the mixed formulation for the Hodge-Laplacian problem. The bilinear form is given by:

$$B((\sigma, u, p), (\tau, v, q)) = \langle \sigma, \tau \rangle - \langle u, d\tau \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle + \langle u, q \rangle.$$

In the continuous case the bilinear form acts on $(V^1 \times V^2 \times \mathfrak{H}^2)^2$ and on $(V_h^1 \times V_h^2 \times \mathfrak{H}_h^2)^2$ in the discrete case. Let $(\sigma, u, p) \in V^1 \times V^2 \times \mathfrak{H}^2$ be such that $\forall (\tau, v, q) \in V^1 \times V^2 \times \mathfrak{H}^2$, $B((\sigma, u, p), (\tau, v, q)) = (f, v)$, and let $(\sigma_h, u_h, p_h) \in V_h^1 \times V_h^2 \times \mathfrak{H}_h^2$ be such that $\forall (\tau, v, q) \in V_h^1 \times V_h^2 \times \mathfrak{H}_h^2$, $B((\sigma_h, u_h, p_h), (\tau, v, q)) = (f, v)$. Then [50, Theorem 5.6 p. 62] gives the error estimate $\|p - p_h\| \lesssim E(p) + \mu_0 E(d\sigma)$. We have $P_{\mathfrak{H}} f = p = 0$, $P_{\mathfrak{H}_h} f = p_h$ and $E(p) = 0$. The well-posedness of the Hodge-Laplacian problem gives:

$$\|P_{\mathfrak{H}_h} f\| = \|p - p_h\| \lesssim 0 + \mu_0 E(d\sigma) \lesssim \mu_0 \|f\|.$$

□

Theorem 3.24. *For $z = (z_2, z_3) \in D_0$, let $z_h = (z_{2h}, z_{3h}) := K_{0h}P_h L_0' z + P_{\mathfrak{H}_h} P_{\mathfrak{H}} z$. It holds:*

$$\|z - z_h\| \lesssim \alpha_0 \|L_0' z\|, \ \|d(z - z_h)\| + \|d^* z_2 - d_h^* z_{2h}\| \lesssim \eta_0 \|L_0' z\|,$$

$$\|P_h(dd^* z_2 + d^* z_3) - (dd_h^* z_{2h} + d_h^* z_{3h})\| \leq \mu_0 \|L_0' z\|.$$

Proof. The same proof as [45, Theorem 5.2] works. Starting from

$$\begin{aligned} z - z_h &= (P_{\mathfrak{H}^\perp} z - P_{\mathfrak{H}_h^\perp} z_h) + (P_{\mathfrak{H}} z - P_{\mathfrak{H}_h} z_h) \\ &= (K_0 - K_{0h} P_h) L_0' z + (I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} z, \end{aligned}$$

$P_{\mathfrak{H}_h} P_{\mathfrak{H}} = P_{\mathfrak{Z}_h} P_{\mathfrak{H}}$ and $\pi_h \mathfrak{Z} \subset \mathfrak{Z}_h$ we infer:

$$\|(I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} z\| \leq \|(I - \pi_h) P_{\mathfrak{H}} z\| \leq \mu_0 \|z\| \lesssim \alpha_0 \|L_0' z\|.$$

The first two estimates follow from Lemma 3.22. For the last estimate, (3.33) gives

$$dd_h^* z_{2h} + d_h^* z_{3h} = (L_{0h}')_2 z_h = (L_{0h}' K_{0h} P_h)_2 L_0' z = (P_{\mathfrak{B}_h} + P_{\mathfrak{B}_h^*}) P_h (L_0')_2 z,$$

$$\begin{aligned} \|P_h(dd^* z_2 + d^* z_3) - (dd_h^* z_{2h} + d_h^* z_{3h})\| &= \|(I - (P_{\mathfrak{B}_h} + P_{\mathfrak{B}_h^*})) P_h (L_0')_2 z\| \\ &= \|P_{\mathfrak{H}_h} (L_0')_2 z\|. \end{aligned}$$

We conclude with Lemma 3.23. □

Given $\mathbf{u} \in V_h$, we define $g := (u_2, P_{\mathfrak{H}^\perp} u_3)$, $z := Kg$, $\xi := -(d^* + l_3^*) z_2$ and $\mathbf{z} := (\xi, z, P_{\mathfrak{H}} u_3)$.

Theorem 3.25. *There is $\mathbf{z}_h \in V_h$ such that $\forall \omega \in V_h$, $\|\mathbf{z}_h\|_V \lesssim \|\mathbf{z}\|_V$ uniformly in h and $|B(\omega, \mathbf{z} - \mathbf{z}_h)| \leq \epsilon_h \|\omega\|_V \|\mathbf{u}\|$, where $\epsilon_h \to 0$ as $h \to 0$.*

Proof. Let $z_h = K_{0h} P_h L_0' z + P_{\mathfrak{H}_h} P_{\mathfrak{H}} z$, $\xi_h = -d_h^* z_{2h} - \Pi_h l_3^* z_2$. Theorem 3.24 gives

$$\begin{gathered} \|z - z_h\| \lesssim \alpha_0 \|g\|, \ \|d(z - z_h)\| \lesssim \eta_0 \|g\|, \ \|d^* z_2 - d_h^* z_{2h}\| \lesssim \eta_0 \|g\|, \\ \|\xi - \xi_h\| \leq \|d^* z_2 - d_h^* z_{2h}\| + \|(I - \Pi_h) l_3^* z_2\|. \end{gathered} \tag{3.34}$$

Lemma 3.20 and the boundedness of $d\, l_3^*\, K$ give:

$$\|(I - \Pi_h) l_3^* z_2\| \lesssim \|(I - \pi_h) l_3^* z_2\| + \eta_0' \|d l_3^* z_2\| \lesssim (\eta + \eta_0') \|g\|.$$

Finally, since $\forall \omega_2 \in V_h^2 \subset V^2$ $\langle d\omega_2, z_3 \rangle = \langle \omega_2, d^* z_3 \rangle$ and $\langle d\omega_2, z_{3h} \rangle = \langle \omega_2, d_h^* z_{3h} \rangle$, we have $\forall \omega \in V_h$:

$$\begin{aligned} &|B(\omega, (\xi - \xi_h, z_2 - z_{2h}, z_3 - z_{3h}, P_{\mathfrak{H}} u_3 - P_{\mathfrak{H}_h} P_{\mathfrak{H}} u_3))| \\ &\quad = |\langle \omega_1, \xi - \xi_h \rangle - \langle \omega_2, d(\xi - \xi_h) \rangle + \langle (\nu d + l_3) \omega_1, z_2 - z_{2h} \rangle \\ &\qquad - \langle \omega_3, d(z_2 - z_{2h}) \rangle + \langle l_5 \omega_2, z_2 - z_{2h} \rangle \\ &\qquad + \langle d\omega_2, z_3 - z_{3h} \rangle + \langle \omega_p, z_3 - z_{3h} \rangle + \langle \omega_3, P_{\mathfrak{H}} u_3 - P_{\mathfrak{H}_h} P_{\mathfrak{H}} u_3 \rangle| \\ &\quad \lesssim \|\omega\|_V (\|\xi - \xi_h\| + 2\|z_2 - z_{2h}\| + \|d(z_2 - z_{2h})\| \\ &\qquad + \|z_3 - z_{3h}\| + \|(I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} u_3\|) + |\langle \omega_2, P_h d^* z_3 - d_h^* z_{3h} - P_h d(\xi - \xi_h) \rangle| \\ &\quad \lesssim \|\omega\|_V \left[(\eta + 2\eta_0 + 2\alpha_0 + \eta_0' + \mu_0) \|\mathbf{u}\| + \|P_h d^* z_3 - d_h^* z_{3h} - P_h d(\xi - \xi_h)\| \right]. \end{aligned}$$

Since η, η_0, η_0', μ_0 and α_0 all converge toward 0 when $h \to 0$, the only thing left to prove is that $\|P_h d^* z_3 - d_h^* z_{3h} - P_h d(\xi - \xi_h)\| \lesssim \epsilon \|\mathbf{u}\|$ where $\epsilon \to 0$ when $h \to 0$. To do so we start from Theorem 3.24 and expand:

$$\|P_h d^* z_3 - d_h^* z_{3h} + P_h dd^* z_2 - dd_h^* z_{2h}\| \lesssim \mu_0 \|g\|,$$

$$-d(\xi - \xi_h) = d(d^* + l_3^*) z_2 - dd_h^* z_{2h} - d\Pi_h l_3^* z_2 = dd^* z_2 - dd_h^* z_{2h} + d(I - \Pi_h) l_3^* z_2.$$

We conclude with Lemma 3.20 since we can find a bounded cochain projection π_h such that $\pi_h d = P_{\mathfrak{B}_h} d$ (see [24, Theorem 3.7]) so

$$\|d(I - \Pi_h) l_3^* z_2\| = \|(I - P_{\mathfrak{B}_h}) d l_3^* z_2\| \lesssim \|(I - \pi_h) d l_3^* z_2\| \leq \eta \|g\|.$$

□

Lemma 3.26. *For all $\mathbf{u} \in V_h$ and $\mathbf{z} \in V$ defined in Theorem 3.25, there exists $c > 0$ independent of h and $\sigma \in V_h$ such that $\|\sigma\|_V \lesssim \|\mathbf{u}\|_V$ and $B(\mathbf{u}, \mathbf{z} + \sigma) \geq c \|\mathbf{u}\|_V^2$.*

Proof. Starting from Lemma 3.17, we construct σ in the same way as we did in Lemma 3.18 in the continuous case. We just have to correct the harmonic part adding

$$B(\omega, (0, 0, -P_{\mathfrak{H}_h} z_3, 0)) = -\langle \omega_p, z_3 \rangle.$$

□

Theorem 3.27. *There are two positive constants h_0 and C_0 such that for all $h \in (0, h_0]$, there exists a unique $\mathbf{u} \in V_h$ such that $\forall \mathbf{v} \in V_h$, $B(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$. Moreover we have $\|\mathbf{u}\|_V \leq C_0 \|\mathbf{f}\|$.*

Proof. For $\mathbf{u} \in V_h$, Theorem 3.25 and Lemma 3.26 give $\mathbf{z} \in V$, $\mathbf{z}_h, \sigma \in V_h$ with $\|\sigma\|_V \lesssim \|\mathbf{u}\|_V$ and two constants b and c independent of h such that:

$$|B(\mathbf{u}, \mathbf{z} + \sigma)| \geq c \|\mathbf{u}\|_V^2, \quad B(\mathbf{u}, \mathbf{z}_h - \mathbf{z}) \leq \epsilon_h b \|\mathbf{u}\|_V^2.$$

Combining the two with a triangle inequality readily gives:

$$\begin{aligned} |B(\mathbf{u}, \sigma + \mathbf{z}_h)| &= |B(\mathbf{u}, \mathbf{z} + \sigma) + B(\mathbf{u}, \mathbf{z}_h - \mathbf{z})| \\ &\geq |B(\mathbf{u}, \mathbf{z} + \sigma)| - |B(\mathbf{u}, \mathbf{z}_h - \mathbf{z})| \\ &\geq c \|\mathbf{u}\|_V^2 - \epsilon_h b \|\mathbf{u}\|_V^2 \geq (c - \epsilon_h b) \|\mathbf{u}\|_V^2. \end{aligned}$$

Since $\epsilon_h \to 0$ as $h \to 0$ we can find h_0 such that $\forall h \in (0, h_0]$, $c - \epsilon_h b \geq c - \epsilon_{h_0} b > 0$.

By Theorem 3.25 and from the expression of σ we find:

$$\|\sigma + \mathbf{z}_h\|_V \lesssim \|\mathbf{u}\|_V + \|\mathbf{z}\|_V \lesssim \|\mathbf{u}\|_V.$$

This ends the proof since V_h is of finite dimension. □

Corollary 3.28. *If assumption* (3.23) *holds then for* $h \le h_0$ *given by Theorem 3.27, and for* $\mathbf{u}$ *(resp.* $\mathbf{u}_h$*) the solution of the continuous problem* (3.26) *(resp. of the discrete problem* (3.32)*) it holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_V \lesssim \inf_{v_1 \in V_h^1} \|u_1 - v_1\|_{V^1} + \inf_{v_2 \in V_h^2} \|u_2 - v_2\|_{V^2} + E(u_3) + E(u_p) + \mu_0 E(P_{\mathfrak{B}} u_3).$$

Proof. The same proof as the one of Theorem 3.10 works. □

Remark 3.29. The hidden constant of Corollary 3.28 depends on $\|l_5\|$ which, in the case of problem (3.4), blows up when $\delta t \to 0$. A more subtle analysis is required to make explicit the dependency of the error on δt. For a single time step in the setting of Corollary 3.28 the error will actually decrease when $\delta t \to 0$. We prove an estimate for the error on u_2 in a very general setting in Theorem 3.46.

3.5 Conserved quantities

Lastly we prove that our scheme does indeed verify the properties mentioned in the introduction as well as the regularity assumption (3.23).

3.5.1 Regularity assumptions

Problem (3.4) is a special case of Problem (3.26) taking suitable l_3 and l_5. We prove below that assumption (3.23) is valid if $u^{n-1} \in H^2(\Omega)$ and if the domain is smooth enough to have $H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \subset H^1(\Omega)$, as discussed in [3, Chapter 3.2]. We use the notation $H^1(\Omega)$ both for scalar and for vector fields. First we need the following lemma:

Lemma 3.30. *If* $B \in H^1(\Omega)$ *and* $A \in H^2(\Omega)$ *then* $A \times B \in H^1(\Omega)$ *for a smooth enough domain* Ω *of* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

Proof. We have $H^1(\Omega) \subset L^4(\Omega)$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$ by Sobolev Embedding theorems thus $\forall i, j, k \in \{x, y, z\}$, $\partial_i A_j \in L^4(\Omega)$, $\partial_i B_j \in L^2(\Omega)$. Terms of the form $A_i\, \partial_j B_k$ are the product of a bounded function with a function in $L^2(\Omega)$ and those of the form $B_i\, \partial_j A_k$ are the product of two functions in $L^4(\Omega)$. □

Going back to problem (3.4), we take $V^1 = H(\text{curl}, \Omega)$, $V^2 = H(\text{div}, \Omega)$, $V^3 = L^2(\Omega)$ and $\mathfrak{H}^3 = 0$. Regarding assumption (3.23), K is a bounded isomorphism from $L^2 \times L^2$ to $(H(\text{div}) \cap H_0(\text{curl})) \times H_0^1 \subset H^1 \times H^1$. If we assume $u \in H^2$ and $v \in H^1$ then using the scalar triple product we have $\forall \sigma \in H(\text{curl})$:

$$2\langle l_3 \sigma, v\rangle = \int (\sigma \times u^{n-1}) \cdot v = \int (u^{n-1} \times v) \cdot \sigma = 2\langle \sigma, l_3^* v\rangle.$$

Thus $l_3^* v = \frac{1}{2} u^{n-1} \times v$ on H^1, l_3^* maps H^1 into itself by Lemma 3.30, and

$$l_3^*(K)_2(W) \subset H^1 \subset H(\text{curl}) = V^1.$$

Thus we have $\|(\nabla\times) l_3^*(K)_2\|_{W\to L^2} \lesssim \|(K)_2\|_{W\to H^1}$ and by the boundedness of $\|K\|_{L^2\times L^2\to H^1\times H^1}$ we get the boundedness of $\|dl_3^*(K)_2\|_{W\to L^2}$. Finally we have $\nu dd^*(K)_2 = I - dl_3^*(K)_2 - l_5^*(K)_2 - d^*(K)_3$ as distributions. From the $L^2 \to L^2$ boundedness of the right-hand side we get both $\nu d^*(K)_2(W) \subset H(\text{curl}) = V^1$ and the boundedness of $\|dd^*(K)_2\|_{W\to L^2}$. The same argument applied to $\overline{L}^{-1}$ shows that $d^*(\overline{L}^{-1})_2(W) \subset V^1$. Hence (3.23) is fulfilled.

Remark 3.31. Assuming $u^{n-1} \in H^2(\Omega)$ is very mild as any solution u of (3.21) must have $\nabla \times u \in H(\text{curl}, \Omega)$, $\nabla \cdot u = 0$ thus $\Delta u \in L^2(\Omega)$. Hence by elliptic regularity for Ω smooth enough and if u satisfies appropriate boundary conditions then $u \in H^2$.

3.5.2 Pointwise vanishing divergence and pressure-robustness

The pointwise vanishing divergence is a simple fact that follows from the use of a discrete subcomplex. From (3.4) we have $\forall q_h \in V_h^3$, $\langle \nabla \cdot u_h + \phi_h, q_h \rangle = 0$ with $\nabla \cdot u_h \perp \phi_h$ and $\nabla \cdot u_h \in V_h^3$ by construction. Therefore, taking $q_h = \nabla \cdot u_h$ we have

$$\langle \nabla \cdot u_h + \phi_h, \nabla \cdot u_h \rangle = \|\nabla \cdot u_h\|^2 = 0.$$

This holds even for Condition (3.7) or (3.8) which does not give a complex structure.

A scheme is called pressure-robust ([47, 62, 57]) if only the pressure (and not the velocity) changes when the external forces acting on the system are modified by a gradient. This property is only valid if there are no harmonic 2-forms.

Every vector field $f \in L^2(\Omega)$ can be written as $f = \nabla \times g + \nabla p$ for some fields g and p. In a bounded domain we only have uniqueness with correct boundary conditions on g and p. As long as these boundary conditions match the ones given in the complex, we have by viewing f as a 2-form, $\nabla \times g = P_{\mathfrak{B}} f$ and $\nabla p = P_{\mathfrak{B}^*} f$. Let $\bar{f} \in L^2(\Omega)$ be another source term and write $(\omega_h, u_h, p_h, \phi_h)$ (resp. $(\bar{\omega}_h, \bar{u}_h, \bar{p}_h, \bar{\phi}_h)$) the solution of (3.32) (resp. (3.32)) for the external forces $(f, 0)$ (resp. $(\bar{f}, 0)$).

Theorem 3.32. *If $\exists s \in V^3$ such that $\bar{f} = f + \nabla s$, then $\omega_h = \bar{\omega}_h$ and $u_h = \bar{u}_h$.*

Proof. If there is $s \in V^3$ such that $f + \nabla s = \bar{f}$ then $P_{\mathfrak{B}}(f - \bar{f}) = 0$, hence $\forall g \in V^1, \langle f - \bar{f}, \nabla \times g \rangle = 0$ and in particular

$$\forall g_h \in V_h^1 \subset V^1, \langle f - \bar{f}, \nabla \times g_h \rangle = 0\,.$$

Therefore $P_{\mathfrak{B}_h}(f-\bar{f}) = 0$, and since we assumed that there were no harmonic 2-forms, $P_h(f-\bar{f}) = P_{\mathfrak{B}_h^*}(f-\bar{f})$. Thus we can find $\xi_h \in V_h^3$ such that $\xi_h \perp \mathfrak{H}_h$ and $d_h^*\xi_h = -P_h(f-\bar{f})$. Moreover $(0,0,\xi_h,0)$ verifies: $\forall(\tau_h, v_h, q_h, \chi_h) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h$,

$$\langle 0, \tau_h\rangle - \langle 0, \nabla\times\tau_h\rangle = 0, \quad \langle \nu d0 + l_3 0 + l_5 0, v_h\rangle - \langle \xi_h, \nabla\cdot v_h\rangle) = \langle f-\bar{f}, v_h\rangle,$$
$$\langle \nabla\cdot 0, q_h\rangle + \langle 0, q_h\rangle = 0, \qquad \langle \xi_h, \chi_h\rangle = 0.$$

By linearity and uniqueness of the solution we have $(\omega_h, u_h, p_h, \phi_h) = (\bar{\omega}_h, \bar{u}_h, \bar{p}_h, \bar{\phi}_h) + (0,0,\xi_h,0)$ and $\omega_h = \bar{\omega}_h, u_h = \bar{u}_h$. □

Remark 3.33. Consider a time iterating scheme. At any time step n, the linear maps l_3^n and l_5^n involved in the determination of $(\omega_h^n, u_h^n, p_h^n, \phi_h^n)$ are functions of u_h^{n-1} and ω_h^{n-1} (more precisely $l_3^n = l_3(u_h^{n-1})$ and $l_5^n = l_5(\omega_h^{n-1})$). However, both ω_h and u_h remain unchanged when the external force is modified by a gradient. As long as $u_h^0 = \bar{u}_h^0$ and $\omega_h^0 = \bar{\omega}_h^0$, an immediate recursion shows that $\forall n$, $u_h^n = \bar{u}_h^n$ and $\omega_h^n = \bar{\omega}_h^n$.

The pressure-robustness allows us to remove the pressure from the error estimate.

Theorem 3.34 (Pressure-robust estimate)**.** *Let* $(\omega, u, p, \phi) := (u_1, u_2, u_3, u_p)$ *be the solution of the continuous problem* (3.26) *and* $(\omega_h, u_h, p_h, \phi_h) := (u_{1h}, u_{2h}, u_{3h}, u_{ph})$ *be the solution of the discrete problem* (3.32)*. Then it holds:*

$$\|(u_1 - u_{1h}, u_2 - u_{2h})\|_{V^1\times V^2} \lesssim \inf_{v_1\in V_h^1} \|u_1 - v_1\|_{V^1} + \inf_{v_2\in V_h^2} \|u_2 - v_2\|_{V^2},$$
$$\|\omega-\omega_h\| + \|\nabla\times(\omega-\omega_h)\| + \|u-u_h\| \lesssim \inf_{\chi\in V_h^1} \|\omega-\chi\|_{V^1} + \inf_{v\in V_h^2} \|u-v\|_{V^2}.$$

Proof. Consider an alternative problem with the same l_3 and l_5 as before but with the source term replaced by $\tilde{f}_2 := f_2 - \nabla p$. Let $(\tilde{\omega}, \tilde{u}, \tilde{p}, \tilde{\phi})$ and $(\tilde{\omega}_h, \tilde{u}_h, \tilde{p}_h, \tilde{\phi}_h)$ be respectively the continuous and discrete solution to this alternative problem. By construction, we have $\tilde{p} = 0$, $\tilde{\phi} = 0$. Hence, the estimate of Corollary 3.28 gives:

$$\|\tilde{\omega} - \tilde{\omega}_h\| + \|\nabla\times(\tilde{\omega} - \tilde{\omega_h})\| + \|\tilde{u} - \tilde{u_h}\| \lesssim \inf_{\chi\in V_h^1} \|\tilde{\omega} - \chi\|_{V^1} + \inf_{v\in V_h^2} \|\tilde{u} - v\|_{V^2}.$$

We conclude since by the pressure-robustness we must have:

$$\tilde{\omega} = \omega,\ \tilde{u} = u,\ \tilde{\omega}_h = \omega_h,\ \tilde{u}_h = u_h\ .$$

□

3.6 Numerical simulations

We validate our scheme with three numerical simulations. The first simulation aims to verify the pressure-robustness property. The second is based on an exact and fully 3D solution of the Navier-Stokes equation constructed by Ethier [7]. We use it to check the convergence rate in space, first on a steady problem then on an unsteady problem. The last simulation focuses on Taylor-Couette flow: we seek the critical speed at which Taylor vortices appear. It is based upon [40, 6, 18]. In any case, we took a unit kinematic viscosity and polynomials of degree 2. The divergence is always checked to be pointwise zero up to machine precision. Our codes are written with the FEniCS computing platform, version 2019.1.0 (See `fenicsproject.org` and [29]) and are available at `https://github.com/mlhanot/Navier-Stokes-feec`.

3.6.1 Pressure robustness

We wish to verify the pressure robustness stated in Theorem 3.32, i.e., that if the external forces acting on two flows differ only by a gradient, then only the pressure differs between the flows. We consider the Stokes no-flow problem in a glass (see [56]). The setup is rather simple. The mesh is a cylinder along the z axis of height 2.0, base radius 1.0 and top radius 1.5, and the force f derives from a potential:

$$f := \frac{\nabla\Phi}{\int_\Omega \Phi}, \quad \Phi := z^\gamma$$

for $\gamma = 1, 2, 4, 7$. We start from a fluid at rest and enforce a no slip condition on the whole boundary. In every case we found a velocity equal to zero at the order of the machine precision. This is not trivial as the same test conducted with Taylor-Hood elements (P_2/P_1) gave a significant nonzero velocity. Moreover the error would scale with ν^{-1} (as shown in [56]).

3.6.2 Convergence rate to an exact solution

We have conducted a convergence analysis with an exact solution. The expression for the solution is given by Ethier [7] and depends on two real parameters a and d. It is given by:

$$u := \begin{bmatrix} -a(\exp(ax)\sin(ay+dz) + \exp(az)\cos(ax+dy))\exp(-d^2t) \\ -a(\exp(ay)\sin(az+dx) + \exp(ax)\cos(ay+dz))\exp(-d^2t) \\ -a(\exp(az)\sin(ax+dy) + \exp(ay)\cos(az+dx))\exp(-d^2t) \end{bmatrix}.$$

We have performed two sets of experiments: the first with $a = 2$ and $d = 0$ and the second with $a = 2$ and $d = 1$. The domain consists of a cylinder of height 2 and

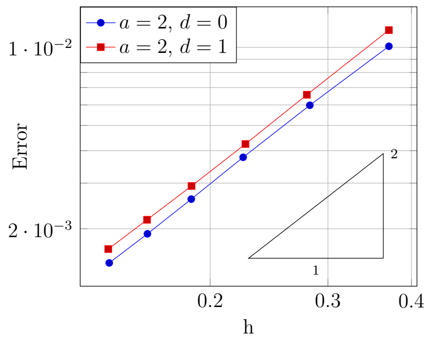


Figure 3.2: Convergence rate of the velocity with relative $H(\text{div})$ error on a log log scale.

radius 1. In the latter case the computation was done for t between 0 and 1 with a time step of 1.0×10^{-3}. We set the velocity to be equal to the velocity of the exact solution on the boundary (without enforcing any boundary condition on the vorticity). For the stationary case ($d = 0$) we start from a fluid at rest, otherwise we start from the exact solution at $t = 0$. We found a rate of convergence in space for the velocity of order 2.0 in both cases, which is in agreement with the theory. Figure 3.2 shows the convergence of the velocity in the relative $H(\text{div})$-norm $\frac{\|u_h - u\|_{H(\text{div})}}{\|u\|_{H(\text{div})}}$ with a log-log scale.

Remark 3.35. We took a time step small enough to neglect the error introduced by the time discretization.

3.6.3 Taylor-Couette flow

This test focuses on Taylor-Couette flow. We follow the work of Gebhardt et al. [40, 6, 18]. The geometry consists in two concentric cylinders of constant radius R_i for the inner and R_o for the outer, rotating at angular velocities Ω_i and Ω_o respectively, and both of height a. The system is closed by two fixed lids at the bottom and top ends. We characterize the system by two geometric parameters: $\eta := R_i/R_o$ and $\Lambda := a/d$ with the gap $d := R_o - R_i$. We also need to define two quantities: the inner Reynolds number $Re_i := \Omega_i R_i d/\nu$ and the outer Reynolds number $Re_o := \Omega_o R_o d/\nu$ where ν is the kinematic viscosity. For an infinite height a, it is a well known fact that, at low speed the flow is steady and fully azimuthal, and that vortices start to form at a critical speed. Since a is finite we expect to see vortices near the lids for speeds way under the critical value (they are however fundamentally different from the Taylor vortices, see [18]).

We compare the results obtained from our code with the reference [6, 18]. The simulations are done starting from a fluid at rest with a no slip boundary condition,

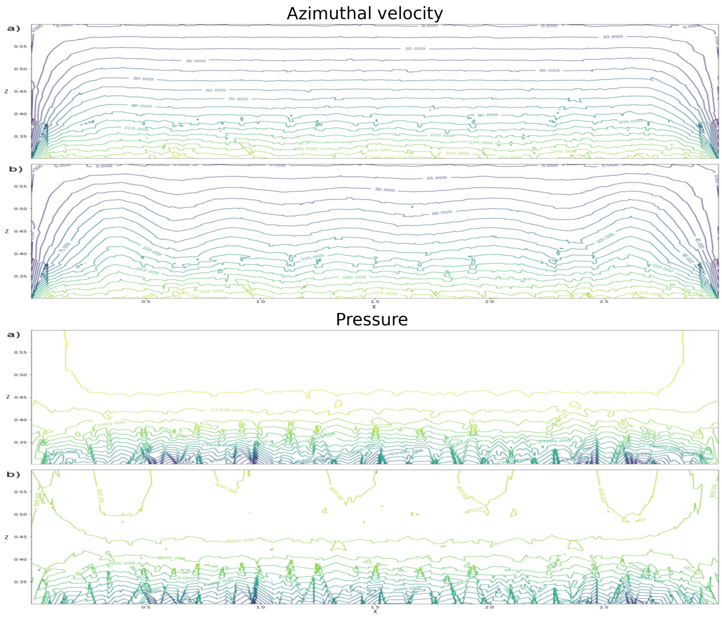


Figure 3.3: Isolines for azimuthal velocity and pressure at $\eta = 0.5$, $Re_o = 0$, $Re_i = 50$ (a) and $Re_i = 72$ (b).

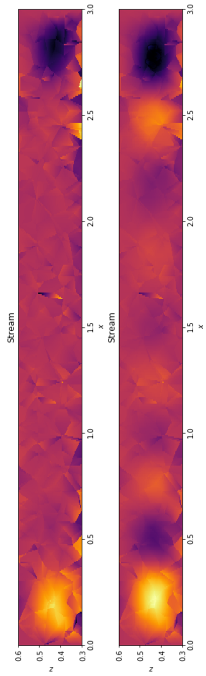


Figure 3.4: Corresponding stream function.

and studied at $t = 0.1$. Typically each simulation is done with a constant time step $\delta t = 0.001$. The time step was decreased for the simulations with the highest velocities. We check the value of Re_i at which the transition occurs for various values of Re_o and η. We display some values taken on the half plane $y = 0, x > 0$ for two values of Re_i at $\eta = 0.5$, $Re_o = 0$. Figure 3.3 shows the azimuthal velocity and the pressure, and Figure 3.4 shows the stream function (the azimuthal component of the vector potential). The transition toward a Taylor-Couette flow occurs suddenly. However, an automatic detection would be difficult to implement, mainly due to interference from the fixed lids. Hence, we obtained an interval rather than a point for the critical value of Re_i (for each value of Re_o and η). Our results are represented by the red rectangles in Figures 3.5 and 3.6. Their height is given by the length of the interval (and their width is arbitrary). The reference data are represented by the black curves. We find very good agreement with the reference, even though we used a much coarser mesh and a smaller aspect ratio Λ of 10 instead of 20, for computational cost reasons.

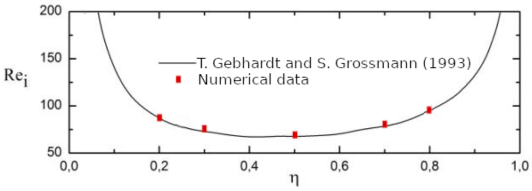


Figure 3.5: Comparison of the critical value of Re_i at $Re_o = 0$ for various η.

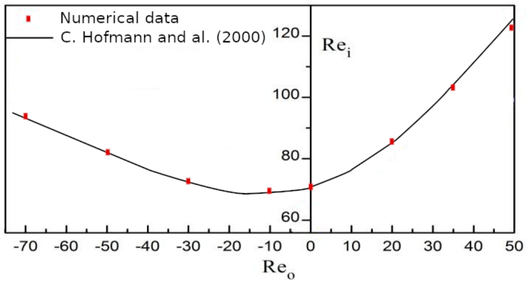


Figure 3.6: Comparison of the critical value of Re_i at $\eta = 0.5$ for various Re_o.

3.A Improved error estimates

This section is dedicated to the proofs of the improved results of Theorem 3.10 which are used in the proof of Lemma 3.22. We will make use of the following lemma (see [24, Lemma 3.12]):

Lemma 3.36. *Let $v_h \in \mathfrak{Z}_h^{k\perp}$ and $v = P_{\mathfrak{B}^*} v_h$. Then*

$$\|v - v_h\| \leq \|(I - \pi_h^k) v\| \leq \eta_0' \|dv_h\|,$$

where η_0' is the scalar given in Lemma 3.20.

Remark 3.37. As a consequence of the proof of [24, Theorem 3.7] we also see that we can take π_h such that for $u \in V^k$, $\|(I - P_{\mathfrak{B}_h}) du\| = \|(I - \pi_h) du\| \lesssim E(du)$.

We can now proceed to the various proofs.

Theorem 3.38. *Let $(u_1, u_2, u_3, \phi_2, \phi_3) \in V^1 \times V^2 \times V^3 \times \mathfrak{H}^2 \times \mathfrak{H}^3$ be the solution of the continuous problem* (3.14) *and let $(u_{1h}, u_{2h}, u_{3h}, \phi_{2h}, \phi_{3h}) \in V_h^1 \times V_h^2 \times V_h^3 \times \mathfrak{H}_h^2 \times \mathfrak{H}_h^3$ be the solution of the discrete problem* (3.18)*. Then the following estimates hold:*

$$\|u_1 - u_{1h}\| \lesssim E(u_1) + \eta_0 E(du_1), \tag{3.35}$$
$$\|du_1 - du_{1h}\| \lesssim E(du_1), \tag{3.36}$$
$$\|u_3 - u_{3h}\| \lesssim E(u_3) + \eta_0'(E(du_1) + E(\phi_2)), \tag{3.37}$$
$$\|P_{\mathfrak{B}_h}(u_2 - u_{2h})\| \lesssim \eta_0 E(u_1) + (\eta_0^2 + \delta_0) E(du_1), \tag{3.38}$$
$$\|P_{\mathfrak{B}_h^*}(u_2 - u_{2h})\| \lesssim E(u_2) + E(du_2), \tag{3.39}$$
$$\|du_2 - du_{2h}\| \lesssim E(du_2). \tag{3.40}$$

For the sake of readability we divide the proof of Theorem 3.38 in the following lemmas.

Lemma 3.39. *The estimate* (3.35) *holds.*

Proof. We use the operator K_0 defined in Section 3.4.3 and recall the decomposition (3.33). We write $u_1 := u_1^1 + u_1^2$ with u_1^1 and u_1^2 defined as follows:

$$u_1 = d^*(K_0)_2(du_1, 0) = d^*(K_0)_2((I - P_{\mathfrak{B}_h})du_1, 0) + d^*(K_0)_2(P_{\mathfrak{B}_h}du_1, 0) := u_1^1 + u_1^2.$$

Let $v_{1h} = \pi_h u_1^2 - u_{1h}$, we have:

$$\begin{aligned} dv_{1h} &= \pi_h du_1^2 - du_{1h} \\ &= \pi_h dd^*(K_0)_2(P_{\mathfrak{B}_h}du_1, 0) - du_{1h} \\ &= \pi_h P_{\mathfrak{B}} P_{\mathfrak{B}_h} du_1 - du_{1h} \\ &= \frac{1}{\nu}(P_{\mathfrak{B}_h} f_2 - P_{\mathfrak{B}_h} f_2) \\ &= 0. \end{aligned}$$

Moreover injecting v_{1h} into (3.14) and (3.18) gives:

$$\langle u_1 - u_{1h}, v_{1h} \rangle = \langle u_2 - u_{2h}, dv_{1h} \rangle = 0.$$

Hence, we have:

$$\begin{aligned} \langle u_1 - u_{1h}, u_1 - u_{1h} \rangle &= \langle u_1 - u_{1h}, \pi_h u_1^2 - u_{1h} \rangle + \langle u_1 - u_{1h}, u_1 - \pi_h u_1^2 \rangle \\ &= 0 + \langle u_1 - u_{1h}, u_1 - \pi_h u_1^2 \rangle \\ &\leq \|u_1 - u_{1h}\| \|u_1 - \pi_h u_1^2\| \end{aligned}$$

and

$$\|u_1 - \pi_h u_1^2\| \leq \|(I - \pi_h)u_1\| + \|\pi_h u_1^1\| \lesssim E(u_1) + \|\pi_h u_1^1\|.$$

It remains to bound $\|\pi_h u_1^1\|$, and since

$$u_1^1 = d^*(K_0)_2(du_1^1, 0) = d^*(K_0)_2((I - P_{\mathfrak{B}_h})du_1, 0),$$

Remark 3.37 gives:

$$\begin{aligned} \langle u_1^1, u_1^1 \rangle &= \langle (K_0)_2(du_1^1, 0), du_1^1 \rangle \\ &= \langle (K_0)_2(du_1^1, 0), (I - P_{\mathfrak{B}_h})du_1 \rangle \\ &= \langle (I - P_{\mathfrak{B}_h})(K_0)_2(du_1^1, 0), (I - P_{\mathfrak{B}_h})du_1 \rangle \\ &\leq \|(I - \pi_h)(K_0)_2(du_1^1, 0)\| \|(I - \pi_h)du_1\| \\ &\lesssim \eta_0 \|u_1^1\| E(du_1). \end{aligned}$$

□

Lemma 3.40. *The estimate* (3.36) *holds.*

Proof. Using the Hodge decomposition of f_2 we have $du_{1h} = P_{\mathfrak{B}_h} f_2 = P_{\mathfrak{B}_h} P_{\mathfrak{B}} f_2 = P_{\mathfrak{B}_h} du_1$. So by Remark 3.37 we have $\|du_1 - du_{1h}\| = \|(I - P_{\mathfrak{B}_h}) du_1\| \lesssim E(du_1)$. □

Lemma 3.41. *The estimate* (3.37) *holds.*

Proof. For any $v_{2h} \in V_h^2$ we have

$$\nu\langle d(u_1 - u_{1h}), v_{2h}\rangle - \langle u_3 - u_{3h}, dv_{2h}\rangle + \langle \phi_2 - \phi_{2h}, v_{2h}\rangle = 0.$$

Since $u_{3h} \perp \mathfrak{H}_h^3$, there is $v_{2h} \in \mathfrak{B}_{2h}^*$ such that $dv_{2h} = P_{\mathfrak{B}_h} u_3 - u_{3h}$. Let $v_2 = P_{\mathfrak{B}^*} v_{2h}$. Using the orthogonality of the Hodge decomposition we have $\langle d(u_1 - u_{1h}), v_2\rangle = 0$ and $\langle \phi_2 - P_{\mathfrak{H}_h}\phi_2, v_2\rangle = 0$ (since $\mathfrak{H}_h \subset \mathfrak{Z}_h \subset \mathfrak{Z} \perp \mathfrak{B}^* \ni v_2$).
Moreover $\langle \phi_2 - \phi_{2h}, v_{2h}\rangle = \langle \phi_2, v_{2h}\rangle = \langle \phi_2 - P_{\mathfrak{H}_h}\phi_2, v_{2h}\rangle$, so it holds:

$$\begin{aligned}\langle u_3 - u_{3h}, P_{\mathfrak{B}_h} u_3 - u_{3h}\rangle &= \langle d(u_1 - u_{1h}), v_{2h}\rangle + \langle \phi_2 - \phi_{2h}, v_{2h}\rangle \\ &= \langle d(u_1 - u_{1h}), v_{2h} - v_2\rangle + \langle \phi_2 - P_{\mathfrak{H}_h}\phi_2, v_{2h} - v_2\rangle \\ &\le (\|d(u_1 - u_{1h})\| + \|\phi_2 - P_{\mathfrak{H}_h}\phi_2\|)\|v_2 - v_{2h}\| \\ &\lesssim (E(du_1) + E(\phi_2))\eta_0'\|P_{\mathfrak{B}_h} u_3 - u_{3h}\|.\end{aligned}$$

We used Lemma 3.36 for the last inequality. Hence we have:

$$\begin{aligned}\|P_{\mathfrak{B}_h} u_3 - u_{3h}\|^2 &= \langle P_{\mathfrak{B}_h} u_3 - u_3, P_{\mathfrak{B}_h} u_3 - u_{3h}\rangle + \langle u_3 - u_{3h}, P_{\mathfrak{B}_h} u_3 - u_{3h}\rangle \\ &\lesssim (\|P_{\mathfrak{B}_h} u_3 - u_3\| + \eta_0'(E(du_1) + E(\phi_2)))\|P_{\mathfrak{B}_h} u_3 - u_{3h}\| \\ &\lesssim (E(u_3) + \eta_0'(E(du_1) + E(\phi_2)))\|P_{\mathfrak{B}_h} u_3 - u_{3h}\|.\end{aligned}$$

Now, since $\|P_h u_3 - P_{\mathfrak{B}_h} u_3\| = \|P_{\mathfrak{H}_h} u_3\| \lesssim \mu_0 E(u_3)$ and $\mu_0 \lesssim 1$ we have:

$$\begin{aligned}\|u_3 - u_{3h}\| &\lesssim \|u_3 - P_h u_3\| + \|P_h u_3 - P_{\mathfrak{B}_h} u_3\| + \|P_{\mathfrak{B}_h} u_3 - u_{3h}\| \\ &\lesssim E(u_3) + \eta_0'(E(du_1) + E(\phi_2)).\end{aligned}$$

□

Lemma 3.42. *The estimate* (3.38) *holds.*

Proof. Let $e = P_{\mathfrak{B}_h}(u_2 - u_{2h})$, $w = (K_0)_2(e, 0)$, $\psi = d^*w$, $w_h = (K_{0h})_2(e, 0)$ and $\psi_h = d_h^* w_h$. First notice that

$$d\psi = dd^*(K_0)_2(e, 0) = P_{\mathfrak{B}} e = e = d\psi_h,$$

so $d\psi \in V_h^2$ and $d\psi = \pi_h d\psi = d\pi_h\psi = d\psi_h$. Moreover $\pi_h\psi - \psi_h \perp \psi - \psi_h$, indeed:

$$\begin{aligned}\langle \pi_h\psi - \psi_h, \psi\rangle &= \langle \pi_h\psi - \psi_h, d^*w\rangle = \langle d\pi_h\psi - d\psi_h, w\rangle = 0, \\ \langle \pi_h\psi - \psi_h, \psi_h\rangle &= \langle \pi_h\psi - \psi_h, d_h^* w_h\rangle = \langle d\pi_h\psi - d\psi_h, w_h\rangle = 0.\end{aligned}$$

This allows us to derive the following bounds:

$$\begin{aligned}\langle \psi - \psi_h, \psi - \psi_h \rangle &= \langle \psi - \pi_h \psi, \psi - \psi_h \rangle + \langle \pi_h \psi - \psi_h, \psi - \psi_h \rangle \\ &= \langle \psi - \pi_h \psi, \psi - \psi_h \rangle \\ &\le \|(I - \pi_h)\psi\| \|\psi - \psi_h\|,\end{aligned}$$

$$\|\psi - \psi_h\| \le \|(I - \pi_h)\psi\| = \|(I - \pi_h) d^*(K_0)_2(e, 0)\| \le \eta_0 \|e\|.$$

Finally, recalling that $du_{1h} = P_{\mathfrak{B}_h} du_1$ (Lemma 3.40), we have:

$$\begin{aligned}\|e\|^2 &= \langle d\psi_h, e \rangle = \langle d\psi_h, u_2 - u_{2h} \rangle = \langle u_1 - u_{1h}, \psi_h \rangle \\ &= \langle u_1 - u_{1h}, \psi_h - \psi \rangle + \langle u_1 - u_{1h}, \psi \rangle \\ &= \langle u_1 - u_{1h}, \psi_h - \psi \rangle + \langle du_1 - du_{1h}, w \rangle \\ &= \langle u_1 - u_{1h}, \psi_h - \psi \rangle + \langle (I - P_{\mathfrak{B}_h}) du_1, (I - P_{\mathfrak{B}_h}) w \rangle \\ &\le \|u_1 - u_{1h}\| \|\psi_h - \psi\| + \|(I - P_{\mathfrak{B}_h}) du_1\| \|(I - P_{\mathfrak{B}_h}) w\| \\ &\lesssim \|u_1 - u_{1h}\| \|\psi_h - \psi\| + \|(I - \pi_h) du_1\| \|(I - \pi_h) w\| \\ &\lesssim \|u_1 - u_{1h}\| \eta_0 \|e\| + E(du_1) \delta_0 \|e\|.\end{aligned}$$

We conclude with the estimate (3.35). □

Lemma 3.43. *The estimate* (3.40) *holds.*

Proof. We know that $du_2 = P_{\mathfrak{B}} f_3$ and $du_{2h} = P_{\mathfrak{B}_h} f_3 = P_{\mathfrak{B}_h} du_2$, hence

$$\|du_2 - du_{2h}\| = \|(I - P_{\mathfrak{B}_h}) du_2\| \lesssim E(du_2).$$

□

Lemma 3.44. *The estimate* (3.39) *holds.*

Proof. Let $v_{2h} = P_{\mathfrak{B}_h^*}(\pi_h u_2 - u_{2h})$. The result follows from triangular inequalities, Poincaré inequalities and (3.40) since:

$$\begin{aligned}\|v_{2h}\| &\le c_p \|d(\pi_h u_2 - u_{2h})\| \\ &\le c_p \|\pi_h du_2 - du_2 + du_2 - du_{2h}\| \\ &\le c_p \|(I - \pi_h) du_2\| + c_p \|du_2 - du_{2h}\| \\ &\lesssim E(du_2).\end{aligned}$$

Hence:

$$\begin{aligned}\|P_{\mathfrak{B}_h^*}(u_2 - u_{2h})\| &\le \|P_{\mathfrak{B}_h^*}(u_2 - \pi_h u_2)\| + \|v_{2h}\| \\ &\lesssim E(u_2) + E(du_2).\end{aligned}$$

□

Corollary 3.45. *Keeping the notation of Theorem 3.38, it holds:*

$$\|u_2 - u_{2h}\| \lesssim E(u_2) + E(du_2) + \eta_0 E(u_1) + (\eta_0^2 + \delta_0) E(du_1).$$

Proof. This is a direct consequence of the estimates (3.38), (3.39) and Lemma 3.23. We simply write:

$$\begin{aligned}\|u_2 - u_{2h}\| &\leq \|u_2 - P_h u_2\| + \|P_{\mathfrak{B}_h}(u_2 - u_{2h})\| + \|P_{\mathfrak{H}_h} u_2\| + \|P_{\mathfrak{B}_h^*}(u_2 - u_{2h})\| \\ &\lesssim E(u_2) + \eta_0 E(u_1) + (\eta_0^2 + \delta_0) E(du_1) + \mu_0 E(u_2) + E(u_2) + E(du_2).\end{aligned}$$

□

3.B Time step independent estimates

When we apply the scheme to solve iteratively a problem some details are not accounted for in the estimate of Corollary 3.28. Indeed $\|l_5\|$ and $\|f_2\|$ depend on δt, and the linear maps l_3 and l_5 are not the same in the continuous and in the discrete problem.

Here we provide an estimate independent of δt, and with two different sets of linear maps for a single step in time. More precisely let (u_1, u_2, u_3, u_p) be the solution of the continuous problem (3.26) for $f_2 := \frac{1}{\delta t} f^c + g$, $f_3 := 0$, $l_3 := l_3^c$, $l_5 := \frac{1}{\delta t} I + l_5^c$. And let $(u_{1h}, u_{2h}, u_{3h}, u_{ph})$ be the solution of the discrete problem (3.32) for $f_2 := \frac{1}{\delta t} f^d + g$, $f_3 := 0$, $l_3 := l_3^d$, $l_5 := \frac{1}{\delta t} I + l_5^d$. For $i \in \{1, 2, 3\}$ we define $e_i := u_i - u_{ih}$ as well as $e_f := f^c - f^d$, $e_{l_3} := l_3^c - l_3^d$ and $e_{l_5} := l_5^c - l_5^d$.

Theorem 3.46. *There exists $\delta t_0 > 0$ depending only on $\|l_3^d\|$, $\|l_5^d\|$ and ν such that $\forall \delta t := \alpha\, \delta t_0$ with $0 < \alpha < 1$,*

$$\begin{aligned}\|P_{\mathfrak{Z}_h} e_2\| \leq \frac{\delta t}{1-\alpha}\Big(&\|l_5^d\| \|(I - P_{\mathfrak{Z}_h}) u_2\| + \|l_3^d\| \|(I - P_h) u_1\| \\ &+ \|e_{l_3}\| \|u_1\| + \|e_{l_5}\| \|u_2\|\Big) + \frac{1}{1-\alpha} \|e_f\|, \\ \|e_2\| \lesssim \|P_{\mathfrak{Z}_h} e_2\| &+ E(u_2).\end{aligned}$$

Proof. Subtracting (3.32) from (3.26) gives:

$$\begin{aligned}\langle e_1, v_{1h}\rangle &= \langle e_2, dv_{1h}\rangle, \\ \langle \nu de_1 + e_{l_3} u_1 + l_3^d e_1 + e_{l_5} u_2 + l_5^d e_2 + \frac{1}{\delta t} e_2, v_{2h}\rangle &= \langle e_3, dv_{2h}\rangle + \langle \frac{1}{\delta t} e_f, v_{2h}\rangle.\end{aligned} \tag{3.41}$$

We take respectively $v_{1h} = P_h e_1$ and $v_{1h} = d_h^* P_{\mathfrak{Z}_h} e_2$ in (3.41) to get:

$$\begin{aligned}\|P_h e_1\|^2 = \langle e_1, P_h e_1\rangle &= \langle e_2, dP_h e_1\rangle = \langle P_{\mathfrak{Z}_h} e_2, dP_h e_1\rangle, \\ \langle dP_h e_1, P_{\mathfrak{Z}_h} e_2\rangle = \langle e_1, d_h^* P_{\mathfrak{Z}_h} e_2\rangle &= \langle e_2, dd_h^* P_{\mathfrak{Z}_h} e_2\rangle = \|d_h^* P_{\mathfrak{Z}_h} e_2\|^2.\end{aligned} \tag{3.42}$$

Taking $v_{2h} = P_{\mathfrak{Z}_h} e_2$ in (3.41) and making use of (3.42), $dP_{\mathfrak{Z}_h} e_2 = 0$ and $|\langle l_3^d P_h e_1, P_{\mathfrak{Z}_h} e_2 \rangle| \le \frac{\nu}{2}\|P_h e_1\|^2 + \frac{\|l_3^d\|^2}{2\nu}\|P_{\mathfrak{Z}_h} e_2\|^2$, we get:

$$\frac{\nu}{2}\|P_h e_1\|^2 + \left(\frac{1}{\delta t} - \frac{\|l_3^d\|^2}{2\nu} - \|l_5^d\|\right)\|P_{\mathfrak{Z}_h} e_2\|^2 \le$$
$$\left(\|l_5^d\|\|(I - P_{\mathfrak{Z}_h})e_2\| + \|l_3^d\|\|(I - P_h)e_1\| + \|e_{l_3}\|\|u_1\| + \|e_{l_5}\|\|u_2\| + \frac{1}{\delta t}\|e_f\|\right)\|P_{\mathfrak{Z}_h} e_2\|. \tag{3.43}$$

Let

$$\delta t_0 := \frac{2\nu}{\|l_3^d\|^2 + 2\nu\|l_5^d\|}.$$

We conclude since it holds $(I - P_h)e_1 = (I - P_h)u_1$, and since $u_{2h} \in \mathfrak{Z}_h$, so $(I - P_{\mathfrak{Z}_h})e_2 = (I - P_{\mathfrak{Z}_h})u_2$. Moreover $u_2 \in \mathfrak{Z}$ and $u_{2h} \in \mathfrak{Z}_h \subset \mathfrak{Z}$, so

$$\|e_2\| = \|P_{\mathfrak{Z}} e_2\| \le \|(P_{\mathfrak{Z}} - P_{\mathfrak{Z}_h})e_2\| + \|P_{\mathfrak{Z}_h} e_2\|,$$

$$\|(P_{\mathfrak{Z}} - P_{\mathfrak{Z}_h})e_2\| = \|(P_{\mathfrak{Z}} - P_{\mathfrak{Z}_h})u_2\| \lesssim \|(I - \pi_h)P_{\mathfrak{Z}} u_2\| \lesssim E(P_{\mathfrak{Z}} u_2) = E(u_2).$$

□

Chapitre 4

An arbitrary-order fully discrete Stokes complex on general polyhedral meshes.

Abstract

In this paper we present an arbitrary-order fully discrete Stokes complex on general polyhedral meshes. We enrich the fully discrete de Rham complex with the addition of a full gradient operator defined on vector fields and fitting into the complex. We show a complete set of results on the novelties of this complex: exactness properties, uniform Poincaré inequalities and primal and adjoint consistency. The Stokes complex is especially well suited for problem involving Jacobian, divergence and curl, like the Stokes problem or magnetohydrodynamic systems. The framework developed here eases the design and analysis of schemes for such problems. Schemes built that way are nonconforming and benefit from the exactness of the complex. We illustrate with the design and study of a scheme solving the Stokes equations and validate the convergence rates with various numerical tests.

4.1 Introduction.

The exactness of the divergence free condition plays an important role in the numerical resolution of incompressible fluid equations, see [48] for a detailed review. This kind of conservation requires the discrete spaces to reproduce relevant algebraic properties of the continuous spaces. Let Ω be a domain of $\mathbb{R}^3$. This exactness can be expressed by the following differential complex:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\text{curl}, \Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\}. \tag{4.1}$$

Many discrete counterparts of the complex (4.1) have been developed. See for example [50] which provides a thorough exposition of conforming finite element complexes, [43, 53, 54] for the elaboration of complexes using the Virtual Element Method (VEM), or [67] for a fully discrete complex and a good exposition of other methods. Although many partial differential equations can be expressed using the de Rham complex (4.1), the lack of smoothness causes issues for some equations, in particular for the Stokes equations (see [27]). So a smoother variant more suited to the Stokes equations (hence called Stokes complex) has been considered. In three dimensions the Stokes complex is written:

$$\mathbb{R} \xrightarrow{i_\Omega} H^2(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}^1(\mathrm{curl},\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\}. \tag{4.2}$$

See e.g. [20, 41, 50]. There are other definitions requiring less smoothness on the first spaces, see [69, 63]. The development of discrete counterparts of this smoother complex is much more complicated. See [50, Chapter 8.7] for a history. In two dimensions the first discrete finite element subcomplex was given by Mardal, Tai, and Winther in 2002 [11]. From then on many other improved subcomplexes were designed (see [32, 31, 44]), as well as discrete subcomplexes based on other methods such as VEM [49]. The three dimensional case is even more challenging. The first finite element constructions require some specific triangulations (e.g. in [15, 26]) or add some tangential continuity to the De Rham complex to approximate the Stokes complex (e.g. in [20, 28]). There are conforming finite element constructions such as [41], however they often have drawbacks. Recurrent problems of the earliest constructions can be a large minimal degree and thus numerous unknowns as well as difficulties to enforce Dirichlet boundary conditions. The subject is very active with many recent advances improving on those issues: see for example the constructions of [69, 61], or the more general construction of a framework for the design of finite elements of differential form of high regularity [51, 58]. Another issue of these constructions is that they are frequently constrained to conformal simplicial meshes, which is limiting for some geometries as well as for the possibility of refinement or agglomeration. A construction of the Stokes complex in virtual finite elements on general meshes has also been recently developed (see [63]).

Our construction works on general polyhedral meshes and for arbitrary polynomial degrees. The discrete spaces are not subspaces of the continuous ones, instead they consist of polynomial spaces on the elements of all geometric dimensions: cells, faces, edges and vertices. The discrete differential operators are therefore necessarily different from the continuous operators. They are constructed according to integration by parts formulae and in a sense converge with the discrete spaces to the continuous operators (see the consistency results of Section 4.5). A discretization of the de Rham complex (4.1) has been developed in detail by D. A. Di Pietro and J. Droniou [67]. One can find in the introduction a very complete comparison of the different methods leading to discrete de Rham complexes on polytopal meshes. Our paper is a continuation of [67]: Our construction is based upon it, and we add the necessary basis functions required for the increased smoothness of the Stokes complex. We define and analyze in detail the Jacobian operator while checking its compatibility with the complex.

More precisely we show the exactness of the complex, the existence of uniform Poincaré inequalities and many consistency results as well as a discrete version of the right inverse for the divergence for the discrete norm $\boldsymbol{H}^1$. Finally, we apply this to the Stokes equations: we show the well-posedness, give an error estimate and find an optimal convergence rate of order $\mathcal{O}(h^{k+1})$, h being the size of the mesh and $k \geq 0$ the chosen polynomial degree. We also validate numerically every result.

As they are both discretization of the Stokes complex on general polyhedral meshes, it is interesting to compare our construction with the VEM one. Conceptually the VEM [63] uses subspaces of the continuous spaces made of functions which are not explicitly known, but are uniquely defined by a set of degrees of freedom located on objects of all dimensions (on the cells, faces, edges and vertices). This allows to use the continuous differential operators, although it is not necessarily trivial to see that they are computable from the given data. The fully discrete Stokes complex on the other hand does not interpret the elements of the discrete spaces as globally defined functions. It uses the same kind of degrees of freedom, but they correspond to distinct functions defined on different domains (e.g. defined on an edge or on a face), and are related only through the discrete differential operators. Despite the different approaches many similarities can be found between the VEM and fully discrete methods. For example, this has be done extensively for the Stokes problem on discrete counterparts of the complex (4.1) in [70]. Although it would be interesting to develop them, there are not yet such deep "bridges" between the two discrete Stokes complexes. It would surely require some adjustments of the scheme to obtain such an equivalence and it is a perspective of research. A first major difference is in the dimension of the spaces of the sequence: while the last two (the discrete $\boldsymbol{H}^1(\Omega)$ and $L^2(\Omega)$ spaces) are of similar size (noticing that the notation for degree is shifted by $+1$ with respect to [63], and without considering the reduced spaces or the possible serendipity procedures [70, Section 6.3]), the first two spaces differ more clearly. Indeed, the VEM complex is focused on the approximation properties of the end of the sequence, and uses small spaces for the beginning (e.g. the only degrees of freedom in the first space are the values at the vertices). On the other hand we want to preserve similar approximation properties throughout the sequence, and to provide consistency results for each space, thus we use spaces with at least enough degrees of freedom to reconstruct polynomials of the given order. This has no impact on the

Stokes problem discussed in [63] and in Section 4.6 since it uses only the last two spaces of the sequence.

A more fundamental difference comes from the use of conformal spaces in the VEM compared to the fully discrete spaces used in our construction. We have to be careful when we use fully discrete spaces because some common consequences of using a complex are not valid anymore. For example, in the context of the Stokes and Navier-Stokes equations, using a discrete complex will give an orthogonal decomposition of the velocity space between the gradients and the divergence-free fields, and ensures that the divergence of any element of the velocity space belongs to the pressure space. For conforming methods such as [63] the latter property gives solutions for the velocity which are exactly divergence free, and the former translates to discrete velocities unaffected by the addition of a gradient in the external forces. This feature known as pressure-robustness is not specific to the use of discrete complexes, and is often found in structure preserving methods. See for example [48] for a review. However this does not hold for non conforming methods such as ours: although we still have the orthogonal decomposition and a discrete divergence operator mapping the velocity space onto the pressure space, we do not have the pressure robustness-property. Indeed the interpolate of a continuous gradient into the discrete space does not fit in the discrete orthogonal decomposition of the velocity space, i.e. it is not orthogonal to the set of discretely divergence-free functions.

The interpolators of our complex are nonetheless chosen to preserve the structure of the continuous spaces: they commute with the continuous and discrete differential operators (see Lemma 4.26). This is the key point to build a pressure-robust method for the Stokes problem on a fully discrete counterpart of the complex (4.1) in [70] which seeks a pressure in the discrete H^1 space and a velocity in the discrete $\boldsymbol{H}(\mathbf{curl}, \Omega)$ space (unfortunately the lack of regularity of the complex (4.1) does not allow to prescribe all components of the velocity on the boundary). Notice that the velocity must be sought in the second space of the complex ($\boldsymbol{H}(\mathbf{curl}, \Omega)$) instead of the third ($\boldsymbol{H}(\mathrm{div}, \Omega)$). Likewise we can show that a discretization on our complex involving a **curl** operator such as a vorticity-velocity-pressure formulation of the Stokes problem (where the vorticity is sought in the discrete $\boldsymbol{H}^1(\mathbf{curl}, \Omega)$ space, the velocity in the discrete $\boldsymbol{H}^1(\Omega)$ space and the pressure in the discrete $L^2(\Omega)$ space) benefits of a "velocity-robustness" property (the discrete pressure is unaffected by the addition of a solenoidal vector field). Hence on a fully discrete complex the potential robustness property comes from the differential operator going into the space rather than the operator defined on it (from the **curl** operator rather than the div operator on the $\boldsymbol{H}^1(\Omega)$ space).

In incompressible fluid dynamics since the velocity is much more involved in the dynamics than the pressure, the "velocity-robustness" property is more difficult to preserve than the pressure-robustness. For example the velocity-robustness will not hold for the velocity-pressure formulation of the Stokes problem presented in Section 4.6 because of the use of the discrete Laplacian instead of the discrete **curl** operator. A interesting prospect would be to considere a scheme for the (Navier-)Stokes equations seeking velocity in the discrete $\boldsymbol{H}^1(\mathbf{curl}, \Omega)$ space since this would give a pressure-robust scheme.

Besides its relationship with the continuous spaces, the discrete Stokes complex enjoys the properties of being a complex. In practice, the algebraic decomposition given by the discrete sequence (4.23) (which is exact when the domain has the correct topology by Theorem 4.28) greatly eases the analysis of schemes (especially for recovering counterparts of properties holding in the continuous spaces, see for example the proof of Lemma 4.34), as well as the design of schemes for multiphysics systems such as magnetohydrodynamic systems.

The remaining of the paper is organized as follows. In Section 4.2 we introduce the general setting. We define the discrete spaces and operators (interpolators, differential operators and norms) in Section 4.3. In Section 4.4 we show that our construction is indeed a complex which is exact given a correct topology. In Section 4.5 we establish consistency properties, including primal and dual consistencies. The Stokes equations are defined and analyzed in Section 4.6. We display our numerical results in Section 4.7. Finally we prove technical propositions in the appendices: on polynomial spaces in appendix 4.A and on various lifts in appendix 4.B.

4.2 Setting.

This section is dedicated to the introduction of the setting and various notations that will be used throughout the paper. We follow the conventions of [67].

4.2.1 Mesh and orientation.

In the following we consider a connected polyhedral domain $\Omega \subset \mathbb{R}^3$ and keeping the notation of [67], for any set $Y \subset \mathbb{R}^3$, we write $h_Y := \sup\{|\boldsymbol{x} - \boldsymbol{y}| : \boldsymbol{x}, \boldsymbol{y} \in Y\}$ and $|Y|$ its Hausdorff measure. We consider on this domain a mesh sequence $\mathcal{M}_h = \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ parameterized by a positive real parameter $h \in \mathcal{H}$. Here $\mathcal{T}_h$ is a finite collection of open polyhedra such that $\overline{\Omega} = \cup_{T \in \mathcal{T}_h} \overline{T}$ and $h = \max_{T \in \mathcal{T}_h} h_T > 0$, $\mathcal{F}_h$ is the collection of open polygonal faces of the cells, $\mathcal{E}_h$ is the collection of open polygonal edges, and $\mathcal{V}_h$ the collection of vertices. This sequence must be regular in the sense of [59, Definition 1.9] with the regularity constant ρ. For any cell $T \in \mathcal{T}_h$, we write $\mathcal{F}_T$ the set of faces of this cell. Likewise for any face $F \in \mathcal{F}_h$, we write $\mathcal{E}_F$ the set of edges of this face.

We take $k \geq 0$ a fixed polynomial degree. In the following most inequalities hold up to a positive constant. This constant depends only on some parameters, here on the chosen polynomial degree k, on the regularity parameter of the mesh sequence ρ and on the domain Ω.

We denote the inequality up to a positive constant by

$$A \lesssim B$$

meaning there exists $C \in \mathbb{R}_+^*$ depending only on some parameters (here usually only on k, ρ and Ω) such that $A \leq CB$. We also write

$$A \approx B$$

meaning that $A \lesssim B$ and $B \lesssim A$.

For any h, we set the orientation of any face $F \in \mathcal{F}_h$ and any edge $E \in \mathcal{E}_h$ by prescribing a unit normal vector $\boldsymbol{n}_F$ and unit tangent vector $\boldsymbol{t}_E$. For any face $F \in \mathcal{F}_h$ and any $E \in \mathcal{E}_F$ we also define the unit vector $\boldsymbol{n}_{FE}$ normal to E lying in the plane tangent to F, and such that $(\boldsymbol{t}_E, \boldsymbol{n}_{FE}, \boldsymbol{n}_F)$ is right-handed in $\mathbb{R}^3$. To keep track of the relative orientation we define for any $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, $\omega_{TF} \in \{-1, 1\}$ such that $\omega_{TF}\boldsymbol{n}_F$ points out of T, and for any $F \in \mathcal{F}_h$, $E \in \mathcal{E}_F$ we define $\omega_{FE} \in \{-1, 1\}$ such that $\omega_{FE}\boldsymbol{n}_{FE}$ points out of F. We also define $\boldsymbol{n}_\Omega$ as the outward pointing unit normal vector on the boundary $\partial\Omega$. We note by $^\perp$ the rotation of angle $\pi/2$ in the oriented plane F.

4.2.2 Polynomial spaces.

For any entity $X \in \{E, F, T\}$, we denote by $\mathcal{P}^k(X)$ the set of polynomials of total degree at most k on X, by $\boldsymbol{\mathcal{P}}^k(X)$ the set of tangential-valued polynomials on X, by $\mathcal{P}^k(X; \mathbb{R}^3)$ the set of $\mathbb{R}^3$-valued polynomials on X, and by $(\boldsymbol{\mathcal{P}}^k(X)^\intercal)^3$ the set of triples of polynomials on X forming the rows of a matrix valued polynomial. We use the conventions $\mathcal{P}^{-1}(X) := \{0\}$ and $\mathcal{P}^{0,k}(X) := \{P \in \mathcal{P}^k(X) \,:\, \int_X P = 0\}$. We also define the broken polynomial space

$$\mathcal{P}^k(\mathcal{X}_h) := \{P_h \in L^2(\mathcal{X}_h) \,:\, \forall X \in \mathcal{X}_h, P_{h|X} \in \mathcal{P}^k(X)\}, \tag{4.3}$$

as well as its continuous counterpart

$$\mathcal{P}_c^k(\mathcal{X}_h) := \{P_h \in C^0(\mathcal{X}_h) \,:\, \forall X \in \mathcal{X}_h, P_{h|X} \in \mathcal{P}^k(X)\}. \tag{4.4}$$

Remark 4.1. Continuous polynomials can be characterized by their values at the interface and their lower order moments on the elements. An explicit construction is deduced from Lemma 4.49. In the context of edges we can see the isomorphism between $\mathcal{P}_c^{k+2}(\mathcal{E}_h)$ and $\mathcal{P}^k(\mathcal{E}_h) \times \mathbb{R}^{\mathcal{V}_h}$.

To distinguish operators acting on scalars from those acting on vectors we use the notation **grad** (resp. ∇) for the operator acting on scalar fields (resp. vector fields) and yielding vector fields (resp. tensor fields).

For the sake of readability we quote two lemmas on discrete spaces: [59, Lemma 1.28 and Lemma 1.32] (in a slightly more restrictive setting):

Lemma 4.2 (Discrete inverse inequality)**.** *Let X be an element of $\mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$. Let a positive integer l and a real number $p \in [1, \infty]$ be fixed. Then, the following inequality holds: For all $v \in \mathcal{P}^l(X)$,*

$$\|\nabla v\|_{L^p(X)} \lesssim h_X^{-1} \|v\|_{L^p(X)}, \tag{4.5}$$

with hidden constant depending only on ρ, l and p.

Lemma 4.3 (Discrete trace inequality)**.** *Let $p \in [1, \infty]$ be a fixed real number and $l \geq 0$ be a fixed integer. Then for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$ (resp. $F \in \mathcal{F}_h$), all $F \in \mathcal{F}_h$ (resp. $E \in \mathcal{E}_h$), all $v \in \mathcal{P}^l(F)$,*

$$\|v\|_{L^p(F)} \lesssim h_T^{-\frac{1}{p}} \|v\|_{L^p(T)} \tag{4.6}$$

with hidden constant depending only on ρ, l and p.

We will also use Koszul complements (see [67, Section 2.4]). We consider for any element $T \in \mathcal{T}_h$ a point $\boldsymbol{x}_T$ such that $B(\boldsymbol{x}_T, \rho h_T) \subset T$. Then we define the following subspaces of $\boldsymbol{\mathcal{P}}^k(T)$:

$$\begin{aligned} \boldsymbol{\mathcal{G}}^k(T) &:= \mathbf{grad}\, \mathcal{P}^{k+1}(T), \quad & \boldsymbol{\mathcal{G}}^{c,k}(T) &:= (\boldsymbol{x} - \boldsymbol{x}_T) \times \boldsymbol{\mathcal{P}}^{k-1}(T), \\ \boldsymbol{\mathcal{R}}^k(T) &:= \mathbf{curl}\, \boldsymbol{\mathcal{P}}^{k+1}(T), \quad & \boldsymbol{\mathcal{R}}^{c,k}(T) &:= (\boldsymbol{x} - \boldsymbol{x}_T) \mathcal{P}^{k-1}(T). \end{aligned} \tag{4.7}$$

These spaces are such that:

$$\boldsymbol{\mathcal{P}}^k(T) = \boldsymbol{\mathcal{G}}^k(T) \oplus \boldsymbol{\mathcal{G}}^{c,k}(T) = \boldsymbol{\mathcal{R}}^k(T) \oplus \boldsymbol{\mathcal{R}}^{c,k}(T), \tag{4.8}$$

however the sum is not orthogonal for the L^2 scalar product.

Similarly, in 2 dimensions for any face $F \in \mathcal{F}_h$, we define:

$$\begin{aligned} \boldsymbol{\mathcal{G}}^k(F) &:= \mathbf{grad}\, \mathcal{P}^{k+1}(F), \quad & \boldsymbol{\mathcal{G}}^{c,k}(F) &:= (\boldsymbol{x} - \boldsymbol{x}_F)^{\perp} \mathcal{P}^{k-1}(F), \\ \boldsymbol{\mathcal{R}}^k(F) &:= \mathbf{rot_F}\, \mathcal{P}^{k+1}(F), \quad & \boldsymbol{\mathcal{R}}^{c,k}(F) &:= (\boldsymbol{x} - \boldsymbol{x}_F) \mathcal{P}^{k-1}(F). \end{aligned} \tag{4.9}$$

We also have the following isomorphisms:

$$\mathbf{rot_F} : \mathcal{P}^{0,k}(F) \to \boldsymbol{\mathcal{R}}^{k-1}(F), \tag{4.10}$$

$$\mathrm{div_F} : \boldsymbol{\mathcal{R}}^{c,k}(F) \to \mathcal{P}^{k-1}(F), \qquad \mathrm{div} : \boldsymbol{\mathcal{R}}^{c,k}(T) \to \mathcal{P}^{k-1}(T), \tag{4.11}$$

$$\mathbf{curl} : \boldsymbol{\mathcal{G}}^{c,k}(T) \to \boldsymbol{\mathcal{R}}^{k-1}(T). \tag{4.12}$$

From the discrete inverse inequality (Lemma 4.2) we can deduce that $|||\mathbf{rot_F}||| \lesssim h_F^{-1}$, $|||\mathrm{div_F}||| \lesssim h_F^{-1}$, $|||\mathrm{div}||| \lesssim h_T^{-1}$, $|||\mathbf{curl}||| \lesssim h_T^{-1}$ and from [67, Lemma A.9] that $|||(\mathbf{rot_F})^{-1}||| \lesssim h_F$, $|||(\mathrm{div_F})^{-1}||| \lesssim h_F$, $|||(\mathrm{div})^{-1}||| \lesssim h_T$, $|||(\mathbf{curl})^{-1}||| \lesssim h_T$.

For $X = \{T, F\}$ we define the local spaces of Nedelec and of Raviart-Thomas respectively by:

$$\boldsymbol{\mathcal{N}}^k(X) := \boldsymbol{\mathcal{G}}^{k-1}(X) \oplus \boldsymbol{\mathcal{G}}^{c,k}(X), \quad \boldsymbol{\mathcal{RT}}^k(X) := \boldsymbol{\mathcal{R}}^{k-1}(X) \oplus \boldsymbol{\mathcal{R}}^{c,k}(X). \tag{4.13}$$

These spaces are strictly contained between $\boldsymbol{\mathcal{P}}^{k-1}(X)$ and $\boldsymbol{\mathcal{P}}^k(X)$. Another important property given in [67, Proposition A.8] is that for any cell $T \in \mathcal{T}_h$ (resp. $F \in \mathcal{F}_h$) and any face of this cell $F \in \mathcal{F}_T$ (resp. $E \in \mathcal{E}_F$):

$$\begin{aligned}
\forall \boldsymbol{v}_F \in \boldsymbol{\mathcal{N}}^k(F),\ (\boldsymbol{v}_F)_{|E} \cdot \boldsymbol{t}_E \in \mathcal{P}^{k-1}(E),\\
\forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{RT}}^k(F),\ (\boldsymbol{w}_F)_{|E} \cdot \boldsymbol{n}_{FE} \in \mathcal{P}^{k-1}(E),\\
\forall \boldsymbol{v}_T \in \boldsymbol{\mathcal{N}}^k(T),\ (\boldsymbol{v}_T)_{|E} \cdot \boldsymbol{t}_E \in \mathcal{P}^{k-1}(E),\\
\forall \boldsymbol{w}_T \in \boldsymbol{\mathcal{RT}}^k(T),\ (\boldsymbol{w}_T)_{|F} \cdot \boldsymbol{n}_F \in \mathcal{P}^{k-1}(F),\\
\forall \boldsymbol{v}_T \in \boldsymbol{\mathcal{N}}^k(T),\ (\boldsymbol{v}_T)_{|F} \times \boldsymbol{n}_F \in \boldsymbol{\mathcal{RT}}^k(F).
\end{aligned} \tag{4.14}$$

In order to fix the notation we write

$$(\boldsymbol{\mathcal{R}}^{c,k}(F)^\intercal)^2 = \begin{pmatrix} \boldsymbol{\mathcal{R}}^{c,k}(F)^\intercal \\ \boldsymbol{\mathcal{R}}^{c,k}(F)^\intercal \end{pmatrix},\ (\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3 = \begin{pmatrix} \boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal \\ \boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal \\ \boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal \end{pmatrix}. \tag{4.15}$$

We take differential operators to be acting row-wise on matrix valued functions, and we use the convention

$$\nabla \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix}.$$

In order to define the discrete Jacobian, we need a space similar to $\boldsymbol{\mathcal{RT}}^k(T)$ for matrix valued polynomials such that the divergence operator realizes an isomorphism to $\boldsymbol{\mathcal{N}}^k(T)$. To this end, we define the space $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T)$ by

$$\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) := \{W \in (\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3 \,:\, \mathrm{Tr}\, W = 0\}. \tag{4.16}$$

An explicit description of this space is given by Lemma 4.43. Let us now construct a complement to this space in $(\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3$. We consider the operator $\boldsymbol{P}_{\mathrm{Tr}}{}^k : \mathcal{P}^{0,k}(T) \to (\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3$:

$$\boldsymbol{P}_{\mathrm{Tr}}{}^k := \begin{pmatrix} \mathrm{div}^{-1} \\ \mathrm{div}^{-1} \\ \mathrm{div}^{-1} \end{pmatrix} \circ \mathbf{grad}, \tag{4.17}$$

where div is the isomorphism from $\boldsymbol{\mathcal{R}}^{c,k}(T)$ into $\mathcal{P}^{k-1}(T)$ given by (4.11). Then we define the space:

$$\widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T) := \boldsymbol{P}_{\mathrm{Tr}}{}^{k}\,\mathcal{P}^{0,k}(T). \tag{4.18}$$

Lemma 4.47 shows that the spaces $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T)$ and $\widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T)$ are complementary. A similar construction of the spaces $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F)$ and $\widehat{\boldsymbol{\mathcal{R}}}^{c,k}(F)$ holds in 2 dimensions, and is used in Appendix 4.C.

Remark 4.4. By construction, we have: $\nabla\cdot\widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T) = \nabla\cdot\boldsymbol{P}_{\mathrm{Tr}}{}^{k}\,\mathcal{P}^{0,k}(T) = \mathbf{grad}\,\mathcal{P}^{k}(T) = \boldsymbol{\mathcal{G}}^{k-1}(T)$.

Remark 4.5. These spaces are hierarchical since $\overline{\boldsymbol{\mathcal{R}}}^{c,k} \subset \overline{\boldsymbol{\mathcal{R}}}^{c,k+1}, \widehat{\boldsymbol{\mathcal{R}}}^{c,k} \subset \widehat{\boldsymbol{\mathcal{R}}}^{c,k+1}$.

We define a matrix valued equivalent to the Raviart-Thomas space as follows

$$\overline{\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{T}}}^{k}(T) := \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) \oplus \widehat{\boldsymbol{\mathcal{R}}}^{c,k-1}(T) \oplus (\boldsymbol{\mathcal{R}}^{k-1}(T)^{\intercal})^{3}. \tag{4.19}$$

Remark 4.6. For $q \in \mathcal{P}^{k}(T)$, we have $q\,\mathbf{I}_{3,3} \in \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T) \oplus (\boldsymbol{\mathcal{R}}^{k}(T)^{\intercal})^{3}$. Indeed $\nabla\cdot\boldsymbol{P}_{\mathrm{Tr}}\,q = \mathbf{grad}\,q$ by Definition (4.17) and $\nabla\cdot(q\,\mathbf{I}_{3,3}) = \mathbf{grad}\,q$ so $\nabla\cdot(\boldsymbol{P}_{\mathrm{Tr}}\,q - q\,\mathbf{I}_{3,3}) = 0$ and $\boldsymbol{P}_{\mathrm{Tr}}\,q - q\,\mathbf{I}_{3,3} \in (\boldsymbol{\mathcal{R}}^{k}(T)^{\intercal})^{3}$ by the isomorphisms (4.11) and (4.8).

Remark 4.7 (Topological decomposition). The decomposition (4.19) is topological in the following sense: if $\boldsymbol{v}^{c}_{\overline{\boldsymbol{\mathcal{R}}},T} \in \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T)$, $\boldsymbol{v}^{c}_{\widehat{\boldsymbol{\mathcal{R}}},T} \in \widehat{\boldsymbol{\mathcal{R}}}^{c,k-1}(T)$ and $\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \in (\boldsymbol{\mathcal{R}}^{k-1}(T)^{\intercal})^{3}$ then

$$\left\|\boldsymbol{v}^{c}_{\overline{\boldsymbol{\mathcal{R}}},T}\right\| + \left\|\boldsymbol{v}^{c}_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\| + \left\|\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}\right\| \approx \left\|\boldsymbol{v}^{c}_{\widehat{\boldsymbol{\mathcal{R}}},T} + \boldsymbol{v}^{c}_{\widehat{\boldsymbol{\mathcal{R}}},T} + \boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}\right\|. \tag{4.20}$$

This is the result of Lemma 4.48.

Lemma 4.8. *For $X \in \{F, T\}$, $\nabla\cdot$ is an isomorphism from $\overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(X)$ to $\boldsymbol{\mathcal{G}}^{c,k}(X)$.*

Proof. The proof is given by Lemma 4.45 for $X = T$ in the appendix. The case $X = F$ is far easier and is provable with the same arguments. □

We will often need to view 2-dimensional spaces as subspace of $\mathbb{R}^3$. In particular we introduce two spaces related to the normal plane of an edge and to the tangent plane of a face. They specify the decomposition of the space used, and are an essential ingredient in the definition of the discrete Jacobian (4.62).

Definition 4.9. For any edge $E \in \mathcal{E}_h$ a natural 3-dimensional vector is $\boldsymbol{t}_E$. We can arbitrarily complete it in an orthonormal basis of $\mathbb{R}^3$ $(\boldsymbol{t}_E, \boldsymbol{n}_1, \boldsymbol{n}_2)$. Assume that $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are fixed once and for all on each edge. We define the space

$$\widetilde{\boldsymbol{\mathcal{P}}}^{k}_{n,E}(E) = \{p_1\boldsymbol{n}_1 + p_2\boldsymbol{n}_2 \;:\; p_1, p_2 \in \mathcal{P}^{k}(E)\}. \tag{4.21}$$

Likewise for any face $F \in \mathcal{F}_h$ assume there is a F-dependent fixed basis $(\boldsymbol{n}_F, \boldsymbol{n}_1, \boldsymbol{n}_2)$ of $\mathbb{R}^3$. We define the space

$$\widetilde{\boldsymbol{\mathcal{P}}}^k(F) = \Bigg\{ \sum_{i,j=\{1,2\}} V_{i,j}\, \boldsymbol{n}_i \otimes \boldsymbol{n}_j + \sum_{i=\{1,2\}} w_i\, \boldsymbol{n}_F \otimes \boldsymbol{n}_i \;: \\ \boldsymbol{V} = (V)_{i,j} \in (\boldsymbol{\mathcal{P}}^k(F)^\intercal)^2, \boldsymbol{w} = (w)_i \in \boldsymbol{\mathcal{P}}^k(F) \Big\}\,. \tag{4.22}$$

Owing to the decomposition $(\boldsymbol{\mathcal{P}}^k(F)^\intercal)^2 = \overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) \oplus \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(F) \oplus (\boldsymbol{\mathcal{R}}^k(F)^\intercal)^2$ (from Lemma 4.47 and (4.8)), we will implicitly write $\widetilde{\boldsymbol{\mathcal{P}}}^k(F) = (\overline{\boldsymbol{\mathcal{R}}}^{c,k}(F) \oplus \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(F) \oplus (\boldsymbol{\mathcal{R}}^k(F)^\intercal)^2) \oplus (\boldsymbol{n}_F \otimes \boldsymbol{\mathcal{P}}^k(F))$ to decompose it into its subcomponents. The last direct sum here is L^2-orthogonal, hence this will not cause any ambiguity in the scalar products. The space $\widetilde{\boldsymbol{\mathcal{P}}}^k(F)$ is isomorph to $M_{3,2}(\mathcal{P}^k(F))$. When embedded in the space of 3 by 3 tensor, elements of $\widetilde{\boldsymbol{\mathcal{P}}}^k(F)$ are all orthogonal to $\boldsymbol{n}_F$ on the right.

We use the notation $\pi^k_{X,Y}$ with $X \in \{\mathcal{P}, \boldsymbol{\mathcal{R}}, \dots\}$ and $Y \in \{E, F, T\}$ for the L^2-orthogonal projection into the polynomial space X of degree k. We add the superscript c for the projection into the complementary space (i.e. $\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{R}},T}$ for the projection into $\boldsymbol{\mathcal{R}}^{c,k}(T)$).

4.3 Discrete complex.

We can now define the discrete complex. We start by giving the degree of freedom and the interpolator of the discrete spaces. Then we define the discrete differential operators and give some basic properties on them.

4.3.1 Complex definition.

We define five discrete spaces $\underline{X}^k_{\mathbf{grad},h}$, $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, $\underline{\boldsymbol{X}}^k_{\nabla,h}$, $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h}$ and $\mathcal{P}^k(\mathcal{T}_h)$. Diagram (4.23) summarizes their connection with each other and with their continuous counterpart. Throughout the paper we will use the notations introduced in this section to refers to the components of discrete vectors.

$$
\begin{array}{ccccccccc}
 & & & & \boldsymbol{L}^2(\Omega) & & & & \\
 & & & & \nabla\uparrow & & & & \\
H^2(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{H}^1(\mathrm{curl},\Omega) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}^1(\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & & \\
\downarrow \underline{I}^k_{\textbf{grad},h} & & \downarrow \underline{\boldsymbol{I}}^k_{\textbf{curl},h} & & \downarrow \underline{\boldsymbol{I}}^k_{\nabla,h} & & \downarrow \pi^k_{P,\mathcal{T}_h} & \downarrow \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,h} & \\
\underline{X}^k_{\textbf{grad},h} & \xrightarrow{\underline{\boldsymbol{G}}^k_h} & \underline{\boldsymbol{X}}^k_{\textbf{curl},h} & \xrightarrow{\underline{\boldsymbol{C}}^k_h} & \underline{\boldsymbol{X}}^k_{\nabla,h} & \xrightarrow{\underline{D}^k_h} & \mathcal{P}^k(\mathcal{T}_h) & & \\
 & & & & \downarrow \underline{\nabla}^{k+1}_h & & & & \\
 & & & & \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h} & & & &
\end{array}
\tag{4.23}
$$

Notice that the interpolators (defined in Section 4.3.2) require more smoothness than the spaces shown in (4.23). Some components of differential operators acting on faces and edges are not computable from the mere value of a function on a given face or edge. Hence we need to include some degrees of freedom corresponding to derivatives of the functions. The discrete $H^2(\Omega)$ counterpart is:

$$
\begin{aligned}
\underline{X}^k_{\textbf{grad},h} :=\{\underline{q}_h = ((\boldsymbol{G}_{q,V})_{V\in\mathcal{V}_h}, (q_E, \boldsymbol{G}_{q,E})_{E\in\mathcal{E}_h}, (q_F, G_{q,F})_{F\in\mathcal{F}_h}, (q_T)_{T\in\mathcal{T}_h}) \ :\
&\boldsymbol{G}_{q,V} \in \mathbb{R}^3, \forall V \in \mathcal{V}_h, \\
&q_E \in \mathcal{P}^{k+1}_c(\mathcal{E}_h), \boldsymbol{G}_{q,E} \in \widetilde{\boldsymbol{\mathcal{P}}}^k_{n,E}(E), \forall E \in \mathcal{E}_h, \\
&q_F \in \mathcal{P}^{k-1}(F), G_{q,F} \in \mathcal{P}^{k-1}(F), \forall F \in \mathcal{F}_h, \\
&q_T \in \mathcal{P}^{k-1}(T), \forall T \in \mathcal{T}_h\}.
\end{aligned}
\tag{4.24}
$$

The unknown $\boldsymbol{G}_{q,V}$ corresponds to the gradient of q, and $\boldsymbol{G}_{q,E}$ (resp. $G_{q,F}$) to the components of $\textbf{grad}\, q$ normal to E (resp. F). The discrete $\boldsymbol{H}^1(\mathrm{curl},\Omega)$ is:

$$
\begin{aligned}
\underline{\boldsymbol{X}}^k_{\textbf{curl},h} :=\{\underline{\boldsymbol{v}}_h = ((\boldsymbol{R}_{v,V})_{V\in\mathcal{V}_h}, (\boldsymbol{v}_E, \boldsymbol{R}_{v,E})_{E\in\mathcal{E}_h}, (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},F}, v_F, \boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}, \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F})_{F\in\mathcal{F}_h},& \\
(\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}, \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},T})_{T\in\mathcal{T}_h}) \ :\ &\boldsymbol{R}_{v,V} \in \mathbb{R}^3, \forall V \in \mathcal{V}_h, \\
&\boldsymbol{v}_E \in \mathcal{P}^{k+2}_c(\mathcal{E}_h;\mathbb{R}^3), \boldsymbol{R}_{v,E} \in \mathcal{P}^{k+1}(E;\mathbb{R}^3), \forall E \in \mathcal{E}_h, \\
&\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F), \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{c,k}(F), v_F \in \mathcal{P}^{k-1}(F), \\
&\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^k(F), \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^{c,k}(F), \forall F \in \mathcal{F}_h, \\
&\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \in \boldsymbol{\mathcal{R}}^{k-1}(T), \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},T} \in \boldsymbol{\mathcal{R}}^{c,k}(T), \forall T \in \mathcal{T}_h\}.
\end{aligned}
\tag{4.25}
$$

The unknown $\boldsymbol{R}_{v,V}$ corresponds to the curl of $\boldsymbol{v}$, $\boldsymbol{R}_{v,E}$ to the pieces of $\textbf{curl}\,\boldsymbol{v}$ involving normal derivatives, and $\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}, \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F}$ to the (rotated) normal derivative

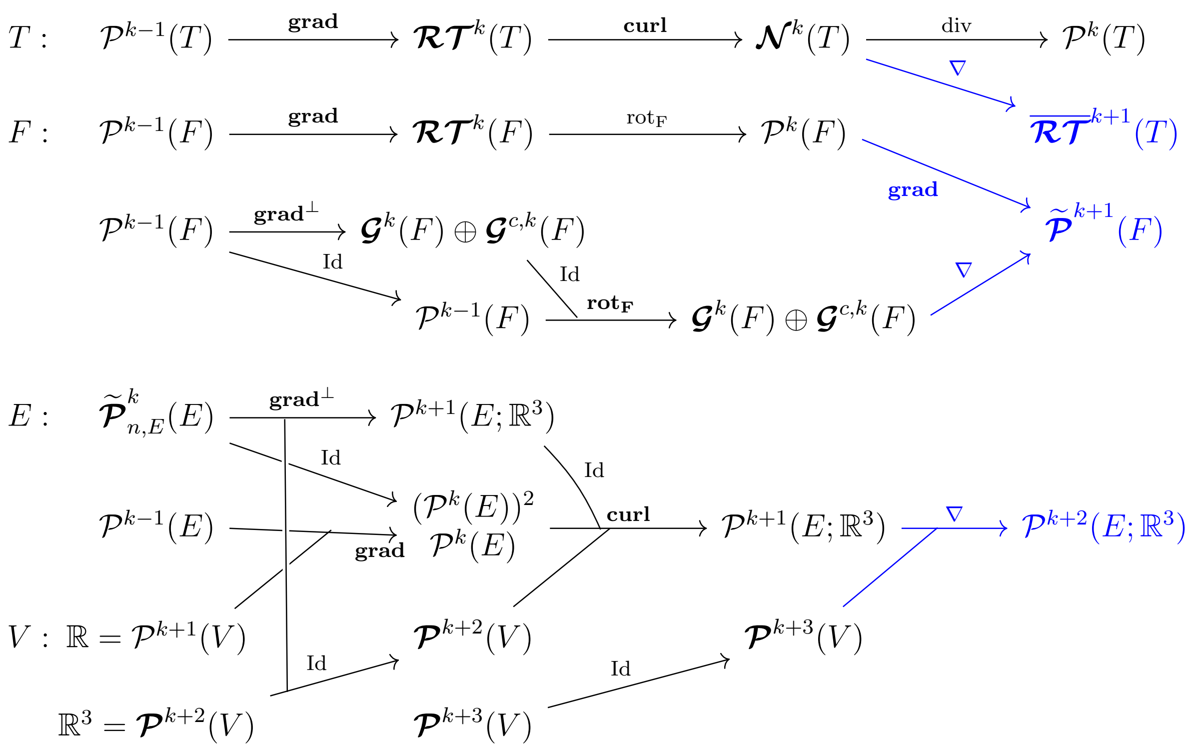


Figure 4.1: Usage of the local degrees of freedom for the discrete differential operators.

of $\boldsymbol{v}$. The discrete $\boldsymbol{H}^1(\Omega)$ is:

$$\begin{aligned}
\underline{\boldsymbol{X}}^k_{\nabla,h} :=\{\underline{\boldsymbol{w}}_h = ((\boldsymbol{w}_E)_{E\in\mathcal{E}_h}, (w_F, \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}, \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F})_{F\in\mathcal{F}_h}, (\boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}, \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T})_{T\in\mathcal{T}_h}) \ : \\
\boldsymbol{w}_E \in \mathcal{P}^{k+3}_c(E;\mathbb{R}^3), \forall E \in \mathcal{E}_h \\
w_F \in \mathcal{P}^k(F), \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^k(F), \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^{c,k}(F), \forall F \in \mathcal{F}_h, \\
\boldsymbol{w}_{\boldsymbol{\mathcal{G}},T} \in \boldsymbol{\mathcal{G}}^{k-1}(T), \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T} \in \boldsymbol{\mathcal{G}}^{c,k}(T), \forall T \in \mathcal{T}_h\},
\end{aligned} \tag{4.26}$$

The last space is not part of the complex, but it is useful for defining a graph norm for the Jacobian (e.g. for the Stokes problem discussed in Section 4.6). The discrete tensor $\boldsymbol{L}^2(\Omega)$ is:

$$\begin{aligned}
\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h} :=\{\underline{\boldsymbol{W}}_h = ((\boldsymbol{W}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{W}_F)_{F\in\mathcal{F}_h}, (\boldsymbol{W}_T)_{T\in\mathcal{T}_h}) \ : \\
\boldsymbol{W}_E \in \mathcal{P}^{k+2}(E;\mathbb{R}^3), \forall E \in \mathcal{E}_h, \\
\boldsymbol{W}_F \in \widetilde{\boldsymbol{\mathcal{P}}}^{k+1}(F), \forall F \in \mathcal{F}_h, \boldsymbol{W}_T \in \overline{\boldsymbol{\mathcal{RT}}}^{k+1}(T), \forall T \in \mathcal{T}_h\}.
\end{aligned} \tag{4.27}$$

Figure 4.1 summarizes the involvement of the various degrees of freedom in the differential operators.

For a given cell T we define the local discrete spaces $\underline{X}^k_{\mathbf{grad},T}$, $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$, $\underline{\boldsymbol{X}}^k_{\nabla,T}$ and $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,T}$ as the restriction of the global one to T, i.e. containing only the components attached to T and those attached to the faces, edges and vertices lying on its boundary. We define in the same way the local discrete spaces attached to a face F or an edge E.

4.3.2 Interpolators.

In this section we define the interpolator linking discrete spaces to their continuous counterpart. Since we project on objects of lower dimension (edges and vertices) we will need a somewhat high smoothness for the functions to interpolate. For a vertex $V \in \mathcal{V}_h$ we define $\boldsymbol{x}_V \in \mathbb{R}^3$ to be its coordinate. The interpolator on the space $\underline{X}^k_{\mathbf{grad},h}$ is defined for any $q \in C^1(\overline{\Omega})$ by

$$\begin{aligned}\underline{I}^k_{\mathbf{grad},h} q = ((\mathbf{grad}\, q(\boldsymbol{x}_V))_{V\in\mathcal{V}_h}, (q_E, \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{t}_E \times (\mathbf{grad}\, q \times \boldsymbol{t}_E)))_{E\in\mathcal{E}_h}, \\ (\pi^{k-1}_{\mathcal{P},F}(q), \pi^{k-1}_{\mathcal{P},F}(\mathbf{grad}(q)\cdot \boldsymbol{n}_F)_{F\in\mathcal{F}_h}, (\pi^{k-1}_{\mathcal{P},T}(q))_{T\in\mathcal{T}_h}),\end{aligned} \tag{4.28}$$

where for any edge $E \in \mathcal{E}_h$, $q_E \in \mathcal{P}^{k+1}_c(\mathcal{E}_h)$ is such that $\pi^{k-1}_{\mathcal{P},E}(q_E) = \pi^{k-1}_{\mathcal{P},E}(q)$ and for any vertex $V \in \mathcal{V}_E$, $q_E(\boldsymbol{x}_V) = q(\boldsymbol{x}_V)$.

The interpolator on the space $\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ is defined for any $\boldsymbol{v} \in \boldsymbol{C}^1(\overline{\Omega})$ by

$$\begin{aligned}\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{v} = ((\mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x}_V))_{V\in\mathcal{V}_h}, (\boldsymbol{v}_E, \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}((\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{t}_E)\boldsymbol{t}_E + \mathbf{grad}(\boldsymbol{v}\cdot\boldsymbol{t}_E)\times\boldsymbol{t}_E))_{E\in\mathcal{E}_h}, \\ (\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{R}},F}(\boldsymbol{v}_{t,F}), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{R}},F}(\boldsymbol{v}_{t,F}), \pi^{k-1}_{\mathcal{P},F}(\boldsymbol{v}\cdot\boldsymbol{n}_F), \\ \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{n}_F\times(\nabla\,\boldsymbol{v}\cdot\boldsymbol{n}_F)), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{n}_F\times(\nabla\,\boldsymbol{v}\cdot\boldsymbol{n}_F)))_{F\in\mathcal{F}_h}, \\ (\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{R}},T}(\boldsymbol{v}), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{R}},T}(\boldsymbol{v}))_{T\in\mathcal{T}_h}),\end{aligned} \tag{4.29}$$

where $\boldsymbol{v}_{t,F}$ is the tangential trace of $\boldsymbol{v}$ on F, and where for any edge $E \in \mathcal{E}_h$, $\boldsymbol{v}_E$ is such that $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{v}_E) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{v})$ and for any vertex $V \in \mathcal{V}_E$, $\boldsymbol{v}_E(\boldsymbol{x}_V) = \boldsymbol{v}(\boldsymbol{x}_V)$. When $\boldsymbol{v} = \mathbf{grad}\, q$ for some function q, $\boldsymbol{n}_F \times (\nabla\,\mathbf{grad}\, q\cdot\boldsymbol{n}_F)$ is just $-\,\mathbf{rot_F}(\partial_{\boldsymbol{n}_F} q)$.

The interpolator on the space $\underline{\boldsymbol{X}}^k_{\nabla,h}$ is defined for any $\boldsymbol{w} \in \boldsymbol{C}^0(\overline{\Omega})$ by

$$\begin{aligned}\underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{w} = ((\boldsymbol{w}_E)_{E\in\mathcal{E}_h}, (\pi^k_{\mathcal{P},F}(\boldsymbol{w}\cdot\boldsymbol{n}_F), \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{w}_{t,F}), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{w}_{t,F}))_{F\in\mathcal{F}_h}, \\ (\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{w}), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{w}))_{T\in\mathcal{T}_h}),\end{aligned} \tag{4.30}$$

where for any edge $E \in \mathcal{E}_h$, $\boldsymbol{w}_E \in \mathcal{P}^{k+3}(E;\mathbb{R}^3)$ is such that $\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{w}_E) = \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{w})$ and for any vertex $V \in \mathcal{V}_E$, $\boldsymbol{w}_E(\boldsymbol{x}_V) = \boldsymbol{w}(\boldsymbol{x}_V)$.

The interpolator on the space $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h}$ is defined for any $\boldsymbol{W} \in \boldsymbol{C}^0(\overline{\Omega})^3$ by

$$\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,h}\boldsymbol{W} = ((\boldsymbol{\pi}^{k+2}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{W}\ \boldsymbol{t}_E))_{E\in\mathcal{E}_h}, (\boldsymbol{\pi}^{k+1}_{\tilde{\boldsymbol{\mathcal{P}}},F}(\boldsymbol{W}))_{F\in\mathcal{F}_h}, (\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\boldsymbol{W}))_{T\in\mathcal{T}_h}). \tag{4.31}$$

The interpolator on the space $\mathcal{P}^k(\mathcal{T}_h)$ is the L^2-orthogonal projection on each cell.

4.3.3 Discrete operators

Gradient.

In the following sections we define the discrete operators starting with the discrete gradient operator $\underline{\boldsymbol{G}}_h^k$. The operator $\underline{\boldsymbol{G}}_h^k$ is the collection of the local discrete operators (4.38) acting on the edges, faces and cells. For any edge $E \in \mathcal{E}_h$ we define the operator $\underline{\boldsymbol{G}}_E^k : \underline{X}_{\mathbf{grad},E}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},E}^k$ such that $\forall \underline{q}_E \in \underline{X}_{\mathbf{grad},E}^k$

$$\underline{\boldsymbol{G}}_E^k \underline{q}_E = ((\mathbf{0})_{V\in\mathcal{V}_E}, \boldsymbol{v}_E, \boldsymbol{v}_E{}' \times \boldsymbol{t}_E), \tag{4.32}$$

where $\boldsymbol{v}_E$ is such that $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k(\boldsymbol{v}_E) = q_E{}'\boldsymbol{t}_E + \boldsymbol{G}_{q,E}$ and $\forall V \in \mathcal{V}_E$, $\boldsymbol{v}_E(\boldsymbol{x}_V) = \boldsymbol{G}_{q,V}$. We write $q_E{}'$ the derivative of q_E along the edge E (oriented by $\boldsymbol{t}_E$).

For any face $F \in \mathcal{F}_h$ we define the operator $\boldsymbol{G}_F^k : \underline{X}_{\mathbf{grad},F}^k \to \boldsymbol{\mathcal{P}}^k(F)$ such that $\forall \underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k$, $\forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{P}}^k(F)$

$$\int_F \boldsymbol{G}_F^k \underline{q}_F \cdot \boldsymbol{w}_F = -\int_F q_F \operatorname{div_F} \boldsymbol{w}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_E \boldsymbol{w}_F \cdot \boldsymbol{n}_{FE}. \tag{4.33}$$

We also define the rotor of the normal gradient $\boldsymbol{RG}_F^k : \underline{X}_{\mathbf{grad},F}^k \to \boldsymbol{\mathcal{P}}^k(F)$ such that $\forall \underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k$, $\forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{P}}^k(F)$

$$\int_F \boldsymbol{RG}_F^k \underline{q}_F \cdot \boldsymbol{w}_F = -\int_F G_{q,F} \operatorname{rot_F} \boldsymbol{w}_F - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{G}_{q,E} \cdot \boldsymbol{n}_F)(\boldsymbol{w}_F \cdot \boldsymbol{t}_E). \tag{4.34}$$

The full operator $\underline{\boldsymbol{G}}_F^k : \underline{X}_{\mathbf{grad},F}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},F}^k$ is defined to be the collection and projection of the local operators. Explicitly for all $\underline{q}_h \in \underline{X}_{\mathbf{grad},F}^k$,

$$\underline{\boldsymbol{G}}_F^k \underline{q}_F = ((\underline{\boldsymbol{G}}_E^k \underline{q}_E)_{E\in\mathcal{E}_F}, (\boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},F}^{k-1}(\boldsymbol{G}_F^k \underline{q}_F), \boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},F}^{c,k}(\boldsymbol{G}_F^k \underline{q}_F), G_{q,F}, \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^k(\boldsymbol{RG}_F^k \underline{q}_F), \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}(\boldsymbol{RG}_F^k \underline{q}_F))). \tag{4.35}$$

The scalar trace $\gamma_{\mathbf{grad},F}^{k+1} : \underline{X}_{\mathbf{grad},F}^k \to \mathcal{P}^{k+1}(F)$ is defined such that $\forall \underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k$, $\forall \boldsymbol{v}_F \in \boldsymbol{\mathcal{R}}^{c,k+2}(F)$,

$$\int_F \gamma_{\mathbf{grad},F}^{k+1} \underline{q}_F \operatorname{div_F} \boldsymbol{v}_F = -\int_F \boldsymbol{G}_F^k \underline{q}_F \cdot \boldsymbol{v}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_E (\boldsymbol{v}_F \cdot \boldsymbol{n}_{FE}). \tag{4.36}$$

Remark 4.10. The relation (4.36) holds for all $\boldsymbol{v}_F \in \boldsymbol{\mathcal{R}}^k(F) \oplus \boldsymbol{\mathcal{R}}^{c,k+2}(F)$. This is the same definition as for the discrete De Rham complex [67].

Remark 4.11. We can also define a tangential trace in the same manner as [67]. It is not required thanks to the choice of norm (4.83) but should be considered to show consistency results for $\underline{X}_{\mathbf{grad},h}^k$.

For any cell $T \in \mathcal{T}_h$ we define the operator $\boldsymbol{G}_T^k : \underline{X}_{\mathbf{grad},T}^k \to \boldsymbol{\mathcal{P}}^k(T)$ such that $\forall \underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k$, $\forall \boldsymbol{w}_T \in \boldsymbol{\mathcal{P}}^k(T)$

$$\int_T \boldsymbol{G}_T^k \underline{q}_T \cdot \boldsymbol{w}_T = -\int q_T \operatorname{div} \boldsymbol{w}_T + \sum_{F \in FT} \omega_{TF} \int_F \gamma_{\mathbf{grad},F}^{k+1} \underline{q}_F \boldsymbol{w}_T \cdot \boldsymbol{n}_F. \tag{4.37}$$

Likewise we define the full operator $\underline{\boldsymbol{G}}_T^k : \underline{X}_{\mathbf{grad},T}^k \to \underline{\boldsymbol{X}}_{\mathbf{curl},T}^k$ for all $\underline{q}_T \in \underline{X}_{\mathbf{grad},T}^k$ by

$$\underline{\boldsymbol{G}}_T^k \underline{q}_T = ((\underline{\boldsymbol{G}}_E^k \underline{q}_E)_{E \in \mathcal{E}_T}, (\boldsymbol{G}_F^k \underline{q}_F)_{F \in \mathcal{F}_T}, (\boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},T}^{k-1}(\boldsymbol{G}_T^k \underline{q}_T), \boldsymbol{\pi}_{\boldsymbol{\mathcal{R}},T}^{c,k}(\boldsymbol{G}_T^k \underline{q}_T)). \tag{4.38}$$

The global operator $\underline{\boldsymbol{G}}_h^k$ is obtained by gathering the local operators $\underline{\boldsymbol{G}}_T^k, T \in \mathcal{T}_h$. Since the interpolators require taking the full gradient even on edges we must consider functions to be defined in a small neighborhood $E \subset \mathcal{O}_E$ open in $\mathbb{R}^3$.

Lemma 4.12 (Consistency properties)**.** *The discrete gradients and trace satisfy the following consistency properties for all $E \in \mathcal{E}_h$, $F \in \mathcal{F}_h$ and $T \in \mathcal{T}_h$:*

$$\underline{\boldsymbol{G}}_E^k(\underline{I}_{\mathbf{grad},E}^k q) = \underline{I}_{\mathbf{curl},E}^k(\mathbf{grad}\, q) \qquad \forall q \in C^2(\mathcal{O}_E) \tag{4.39}$$

$$\boldsymbol{G}_F^k(\underline{I}_{\mathbf{grad},F}^k q) = \mathbf{grad}\, q \qquad \forall q \in \mathcal{P}^{k+1}(F) \tag{4.40}$$

$$\boldsymbol{R}\boldsymbol{G}_F^k(\underline{I}_{\mathbf{grad},F}^k(q\, \boldsymbol{x} \cdot \boldsymbol{n}_F)) = -\,\mathbf{rot_F}\, q \qquad \forall q \in \mathcal{P}^{k+1}(F) \tag{4.41}$$

$$\gamma_{\mathbf{grad},F}^{k+1}(\underline{I}_{\mathbf{grad},F}^k q) = q \qquad \forall q \in \mathcal{P}^{k+1}(F) \tag{4.42}$$

$$\pi_{\mathcal{P},F}^{k-1}(\gamma_{\mathbf{grad},F}^{k+1} \underline{q}_F) = q_F \qquad \forall \underline{q}_F \in \underline{X}_{\mathbf{grad},F}^k \tag{4.43}$$

$$\boldsymbol{G}_T^k(\underline{I}_{\mathbf{grad},T}^k q) = \mathbf{grad}\, q \qquad \forall q \in \mathcal{P}^{k+1}(T) \tag{4.44}$$

Proof. Properties (4.40), (4.42), (4.43) and (4.44) are proven in [67, Lemma 3.3].

Proof of (4.39). The idea is to use integration by parts together with the continuity on vertices to remove the projection. Let $E \in \mathcal{E}_h$, $q \in C^2(\mathcal{O}_E)$, $\underline{\boldsymbol{G}}_E^k(\underline{I}_{\mathbf{grad},E}^k q) := ((\mathbf{0})_{V \in \mathcal{V}_E}, \boldsymbol{v}_E, \boldsymbol{v}_E{}' \times \boldsymbol{t}_E)$, q_E as in (4.28) and $\boldsymbol{w}_E \in \mathcal{P}^{k+2}(E;\mathbb{R}^3)$ such that $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k(\boldsymbol{w}_E) = \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k(\mathbf{grad}\, q)$ and $\boldsymbol{w}_E(\boldsymbol{x}_V) = \mathbf{grad}\, q(\boldsymbol{x}_V)$ for $V \in \mathcal{V}_E$. We have $\mathbf{0} = \mathbf{curl}\,\mathbf{grad}\, q(\boldsymbol{x}_V)$, $\forall V \in \mathcal{V}_E$. We must show that $\boldsymbol{v}_E = \boldsymbol{w}_E$ and that $\boldsymbol{v}_E{}' \times \boldsymbol{t}_E = \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1}(((\mathbf{curl}\,\mathbf{grad}\, q) \cdot \boldsymbol{t}_E)\boldsymbol{t}_E + (\nabla(\mathbf{grad}\, q \cdot \boldsymbol{t}_E) \times \boldsymbol{t}_E)) = \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1}((\nabla(\mathbf{grad}\, q \cdot \boldsymbol{t}_E) \times \boldsymbol{t}_E))$. Take a standard basis $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2)$ such that $\boldsymbol{t}_E = (1,0,0)$ and $\boldsymbol{v}_E := (v_0, v_1, v_2)$. For all $\boldsymbol{r} \in \mathcal{P}^k(E;\mathbb{R}^3)$,

$$\begin{aligned}
\int_E \boldsymbol{v}_E \cdot \boldsymbol{r} &= \int_E q_E{}'\, r_0 + \int_E \pi_{\mathcal{P},E}^k(\partial_1 q)\, r_1 + \pi_{\mathcal{P},E}^k(\partial_2 q)\, r_2 \\
&= -\int_E q_E r_0{}' + [q_E\, r_0] + \int_E \pi_{\mathcal{P},E}^k(\partial_1 q)\, r_1 + \pi_{\mathcal{P},E}^k(\partial_2 q)\, r_2 \\
&= -\int_E q r_0{}' + [q\, r_0] + \int_E \partial_1 q\, r_1 + \partial_2 q\, r_2 \\
&= \int_E \mathbf{grad}\, q \cdot \boldsymbol{r}.
\end{aligned}$$

We use that $q_E(\boldsymbol{x}_V) = q(\boldsymbol{x}_V)$, $V \in \mathcal{V}_E$ and $\pi^{k-1}_{\mathcal{P},E}(q_E) = \pi^{k-1}_{\mathcal{P},E}(q)$ to get the third line. Hence, $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{v}_E) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\mathbf{grad}\, q)$ and since $\boldsymbol{v}_E(\boldsymbol{x}_V) = \mathbf{grad}\, q(\boldsymbol{x}_V)$ we have $\boldsymbol{v}_E = \boldsymbol{w}_E$. We conclude with the same argument for $\boldsymbol{r} \in \mathcal{P}^{k+1}(E;\mathbb{R}^3)$ applied to $\boldsymbol{v}_E{}' \times \boldsymbol{t}_E$.

Proof of (4.41). Let $q \in \mathcal{P}^{k+1}(F)$. Since q does not depend on the coordinate in the $\boldsymbol{n}_F$ direction, we have $\mathbf{grad}(q\ \boldsymbol{x}\cdot\boldsymbol{n}_F)\cdot\boldsymbol{n}_F = q$. The relevant parts of $\underline{I}^k_{\mathbf{grad},F}(q\ \boldsymbol{x}\cdot\boldsymbol{n}_F)$ are $G_{q\,\boldsymbol{x}\cdot\boldsymbol{n}_F,F} = \pi^{k-1}_{\mathcal{P},F}(\mathbf{grad}(q\ \boldsymbol{x}\cdot\boldsymbol{n}_F)\cdot\boldsymbol{n}_F) = \pi^{k-1}_{\mathcal{P},F}(q)$ and $\boldsymbol{G}_{q\,\boldsymbol{x}\cdot\boldsymbol{n}_F,E} = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\mathbf{grad}(q\ \boldsymbol{x}\cdot\boldsymbol{n}_F) - (\mathbf{grad}(q\ \boldsymbol{x}\cdot\boldsymbol{n}_F)\cdot\boldsymbol{t}_E)\boldsymbol{t}_E)$. Hence we have $\boldsymbol{G}_{q\,\boldsymbol{x}\cdot\boldsymbol{n}_F,E}\cdot\boldsymbol{n}_F = \pi^k_{\mathcal{P},E}(q)$ and for all $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}^k(F)$,

$$\begin{aligned}\int_F \boldsymbol{RG}^k_F(\underline{I}^k_{\mathbf{grad},F}(q\ \boldsymbol{x}\cdot\boldsymbol{n}_F))\cdot\boldsymbol{w} &= -\int_F \pi^{k-1}_{\mathcal{P},F}(q)\,\mathrm{rot_F}\,\boldsymbol{w} - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \pi^k_{\mathcal{P},E}(q)\,\boldsymbol{w}\cdot\boldsymbol{t}_E \\ &= -\int_F q\,\mathrm{rot_F}\,\boldsymbol{w} - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E q\,\boldsymbol{w}\cdot\boldsymbol{t}_E \\ &= -\int_F \mathbf{rot_F}\, q\cdot\boldsymbol{w}.\end{aligned}$$

□

We will also need a scalar potential reconstruction on cells $P^{k+1}_{\mathbf{grad},T} : \underline{X}^k_{\mathbf{grad},T} \to \mathcal{P}^{k+1}(T)$. We use the same reconstruction as in [67]. It is defined on a cell $T \in \mathcal{T}_h$ such that for all $\underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}$, and all $\boldsymbol{v}_T \in \boldsymbol{\mathcal{R}}^{c,k+2}(T)$,

$$\int_T P^{k+1}_{\mathbf{grad},T}\underline{q}_T\ \mathrm{div}\,\boldsymbol{v}_T = -\int_T \boldsymbol{G}^k_T\underline{q}_T\cdot\boldsymbol{v}_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \gamma^{k+1}_{\mathbf{grad},F}\underline{q}_F\,\boldsymbol{v}_T\cdot\boldsymbol{n}_F\,. \quad (4.45)$$

The following consistency properties are proven in [67]:

$$P^{k+1}_{\mathbf{grad},T}(\underline{I}^k_{\mathbf{grad},T}q) = q \qquad \forall q \in \mathcal{P}^{k+1}(T), \quad (4.46)$$

$$\pi^{k-1}_{\mathcal{P},T}(P^{k+1}_{\mathbf{grad},T}\underline{q}_T) = q_T \quad \forall \underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}. \quad (4.47)$$

Curl.

The operator $\underline{\boldsymbol{C}}^k_h$ is the collection of the local discrete operators (4.58) acting on the edges and faces. For any edge $E \in \mathcal{E}_h$ we define the operator $\boldsymbol{C}^k_E : \underline{\boldsymbol{X}}^k_{\mathbf{curl},E} \to \mathcal{P}^{k+3}(E;\mathbb{R}^3)$ such that $\forall \underline{\boldsymbol{v}}_E = ((\boldsymbol{R}_{v,V})_{V\in\mathcal{V}_E}, \boldsymbol{v}_E, \boldsymbol{R}_{v,E}) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},E}$

$$\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{C}^k_E\underline{\boldsymbol{v}}_E) = \boldsymbol{R}_{v,E} - \boldsymbol{v}_E{}'\times\boldsymbol{t}_E, \quad \boldsymbol{C}^k_E\underline{\boldsymbol{v}}_E(\boldsymbol{x}_V) = \boldsymbol{R}_{v,V}. \quad (4.48)$$

For any face $F \in \mathcal{F}_h$ we define the operators $C^k_F : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \mathcal{P}^k(F)$ and $\boldsymbol{C}^k_F : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \boldsymbol{\mathcal{P}}^k(F)$ for all $\underline{\boldsymbol{v}}_F = ((\boldsymbol{v}_E, \boldsymbol{R}_{v,E})_{E\in\mathcal{E}_h}, (\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}, \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},F}, v_F, \boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}, \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F})) \in$

$\underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$ such that $\forall r_F \in \mathcal{P}^k(F)$

$$\int_F C^k_F \underline{\boldsymbol{v}}_F r_F = \int_F \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \cdot \mathbf{rot_F}\, r_F - \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v}_E \cdot \boldsymbol{t}_E)\, r_F, \tag{4.49}$$

and $\forall \boldsymbol{r}_F \in \boldsymbol{\mathcal{P}}^k(F)$

$$\int_F \boldsymbol{C}^k_F \underline{\boldsymbol{v}}_F \cdot \boldsymbol{r}_F = \int_F v_F \,\mathrm{rot_F}\, \boldsymbol{r}_F + \sum_{E\in EF} \omega_{FE} \int_E (\boldsymbol{v}_E \cdot \boldsymbol{n}_F)(\boldsymbol{r}_F \cdot \boldsymbol{t}_E). \tag{4.50}$$

The full operator $\underline{\boldsymbol{C}}^k_F : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \underline{\boldsymbol{X}}^k_{\nabla,F}$ is defined as the collection and projection of the local operators. We also add the contribution from the normal derivative contained in $\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}$ and $\boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F}$ that $\boldsymbol{C}^k_F$ lacks. Explicitly for all $\underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}$

$$\underline{\boldsymbol{C}}^k_F \underline{\boldsymbol{v}}_F = ((\boldsymbol{C}^k_E \underline{\boldsymbol{v}}_E)_{E\in\mathcal{E}_F}, (C^k_F \underline{\boldsymbol{v}}_F, \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{C}^k_F \underline{\boldsymbol{v}}_F) + \boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}, \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{C}^k_F \underline{\boldsymbol{v}}_F) + \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F})). \tag{4.51}$$

Lemma 4.13 (Local complex property)**.** *For all $F \in \mathcal{F}_h$ it holds:*

$$\mathrm{Im}\, \underline{\boldsymbol{G}}^k_F \subset \mathrm{Ker}\, \underline{\boldsymbol{C}}^k_F. \tag{4.52}$$

Proof. Let $\underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$, we have to show that $\underline{\boldsymbol{C}}^k_F(\underline{\boldsymbol{G}}^k_F \underline{q}_F) = 0$. We define $\underline{\boldsymbol{v}}_F = \underline{\boldsymbol{G}}^k_F \underline{q}_F$. It is immediate to check for the edges since $\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{C}^k_E \underline{\boldsymbol{v}}_E) = \boldsymbol{R}_{v,E} - \boldsymbol{v}_E{}' \times \boldsymbol{t}_E$ with $\boldsymbol{R}_{v,E} = \boldsymbol{v}_E{}' \times \boldsymbol{t}_E$ and $\boldsymbol{C}^k_E \underline{\boldsymbol{v}}_E(\boldsymbol{x}_V) = \boldsymbol{0}$. Moreover [67, Proposition 3.2] gives $C^k_F \underline{\boldsymbol{v}}_F = 0$. It remains to prove $\boldsymbol{C}^k_F \underline{\boldsymbol{v}}_F = 0$. For any $\boldsymbol{r}_F$ successively in $\boldsymbol{\mathcal{G}}^k(F)$ and $\boldsymbol{\mathcal{G}}^{c,k}(F)$ it is immediate to check that $\int_F \boldsymbol{C}^k_F(\underline{\boldsymbol{G}}^k_F \underline{q}_F) \cdot \boldsymbol{r}_F = 0$ since (4.50) and (4.34) are opposite. □

We define the tangential trace $\gamma^k_{t,\mathrm{rot},F} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},F} \to \boldsymbol{\mathcal{P}}^k(F)$ such that $\forall (r_F, \boldsymbol{w}_F) \in \mathcal{P}^{0,k+1}(F) \times \boldsymbol{\mathcal{R}}^{c,k}(F)$,

$$\int_F \gamma^k_{t,\mathrm{rot},F} \underline{\boldsymbol{v}}_F \cdot (\mathbf{rot_F}\, r_F + \boldsymbol{w}_F) = \int_F C^k_F \underline{\boldsymbol{v}}_F r_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E (\boldsymbol{v}_E \cdot \boldsymbol{t}_E) r_F + \int_F \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},F} \cdot \boldsymbol{w}_F. \tag{4.53}$$

This is almost the same definition as in [67]. As such we have almost the same properties.

Lemma 4.14 (Properties of the tangential trace)**.** *It holds*

$$\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{R}},F}(\gamma^k_{t,\mathrm{rot},F} \underline{\boldsymbol{v}}_F) = \boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \text{ and } \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{R}},F}(\gamma^k_{t,\mathrm{rot},F} \underline{\boldsymbol{v}}_F) = \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},F} \quad \forall \underline{\boldsymbol{v}}_F \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},F}, \tag{4.54}$$

$$\gamma^k_{t,\mathrm{rot},F}(\underline{I}^k_{\mathbf{curl},F} \boldsymbol{v}) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},F} \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{k+1}(F), \tag{4.55}$$

$$\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},F}(\gamma^k_{t,\mathrm{rot},F}(\underline{\boldsymbol{G}}^k_F \underline{q}_F)) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},F}(\boldsymbol{G}^k_F \underline{q}_F) \quad \forall \underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}. \tag{4.56}$$

Proof. The proof of [67, Proposition 3.3] almost works here, the sole difference being for (4.56). Indeed if $\underline{q}_F \in \underline{X}^k_{\mathbf{grad},F}$ and $\underline{\boldsymbol{v}}_F = \underline{\boldsymbol{G}}^k_F \underline{q}_F$ then on the boundary we will only have $\pi^k_{\mathcal{P},E}(\boldsymbol{v}_E \cdot \boldsymbol{t}_E) = q_E{}'$ instead of $\boldsymbol{v}_E \cdot \boldsymbol{t}_E = q_E{}'$. Therefore we must restrict ourselves to test functions in $\mathcal{P}^{0,k}(F) \times \boldsymbol{\mathcal{R}}^{c,k}(F)$ instead of $\mathcal{P}^{0,k+1}(F) \times \boldsymbol{\mathcal{R}}^{c,k}(F)$. This explains the addition of $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{RT}},F}$ since $\mathbf{rot_F}\, \mathcal{P}^{0,k}(F) \oplus \boldsymbol{\mathcal{R}}^{c,k}(F) = \boldsymbol{\mathcal{RT}}^k(F)$. □

For any $T \in \mathcal{T}_h$ we define the operator $\boldsymbol{C}^k_T : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \boldsymbol{\mathcal{P}}^k(T)$ for all $\underline{\boldsymbol{v}}_T = ((\underline{\boldsymbol{v}}_F)_{F\in\mathcal{F}_T}, \boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}, \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},T}) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$ such that $\forall \boldsymbol{r}_T \in \boldsymbol{\mathcal{P}}^k(T)$,

$$\int_T \boldsymbol{C}^k_T \underline{\boldsymbol{v}}_T \cdot \boldsymbol{r}_T = \int_T \boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \cdot \mathbf{curl}\, \boldsymbol{r}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^k_{t,\mathrm{rot},F} \underline{\boldsymbol{v}}_F \cdot (\boldsymbol{r}_T \times \boldsymbol{n}_F). \tag{4.57}$$

The full operator $\underline{\boldsymbol{C}}^k_T : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \underline{\boldsymbol{X}}^k_{\nabla,T}$ is such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$,

$$\underline{\boldsymbol{C}}^k_T \underline{\boldsymbol{v}}_T := ((\boldsymbol{C}^k_E \underline{\boldsymbol{v}}_E)_{E\in\mathcal{E}_T}, (\underline{\boldsymbol{C}}^k_F \underline{\boldsymbol{v}}_F)_{F\in\mathcal{F}_T}, \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{C}^k_T \underline{\boldsymbol{v}}_T), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{C}^k_T \underline{\boldsymbol{v}}_T)). \tag{4.58}$$

We will also need the vector potential reconstruction $\boldsymbol{P}^k_{\mathbf{curl},T} : \underline{\boldsymbol{X}}^k_{\mathbf{curl},T} \to \boldsymbol{\mathcal{P}}^k(T)$ defined in [67, Section 4.2]. The following properties are proven therein:

$$\boldsymbol{P}^k_{\mathbf{curl},T}(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T} \boldsymbol{v}) = \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(T), \tag{4.59}$$

$$\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{R}},T}(\boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T) = \boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \text{ and } \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{R}},T}(\boldsymbol{P}^k_{\mathbf{curl},T} \underline{\boldsymbol{v}}_T) = \boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},T} \quad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}. \tag{4.60}$$

Jacobian.

The operator $\underline{\nabla}^{k+1}_h$ is the collection of the local discrete operators (4.70). For all edge $E \in \mathcal{F}_h$ we define the operator $\nabla^{k+2}_E : \underline{\boldsymbol{X}}^k_{\nabla,E} \to \mathcal{P}^{k+2}(E;\mathbb{R}^3)$ for all $\boldsymbol{w}_E \in \underline{\boldsymbol{X}}^k_{\nabla,E}$ by

$$\nabla^{k+2}_E\, \boldsymbol{w}_E = \boldsymbol{w}_E{}'. \tag{4.61}$$

For each face $F \in \mathcal{F}_h$ we define the operator $\nabla^{k+1}_F : \underline{\boldsymbol{X}}^k_{\nabla,F} \to \widetilde{\boldsymbol{\mathcal{P}}}^{k+1}(F)$ for all $\underline{\boldsymbol{w}}_F = ((\boldsymbol{w}_E)_{E\in\mathcal{E}_h}, w_F, \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}, \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}) \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ such that $\forall \boldsymbol{V}_F = \boldsymbol{V}_{\boldsymbol{\mathcal{P}},\boldsymbol{n}_F} + \boldsymbol{V}^c_{\overline{\boldsymbol{\mathcal{R}}},F} + \boldsymbol{V}^c_{\widehat{\boldsymbol{\mathcal{R}}},F} + \boldsymbol{V}_{\boldsymbol{\mathcal{R}},F} \in \widetilde{\boldsymbol{\mathcal{P}}}^{k+1}(F)$,

$$\begin{aligned}\int_F \nabla^{k+1}_F(\underline{\boldsymbol{w}}_F) : \boldsymbol{V}_F = &- \int_F \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F} \cdot \nabla\cdot(\boldsymbol{V}^c_{\overline{\boldsymbol{\mathcal{R}}},F}) - \int_F \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F} \cdot \nabla\cdot(\boldsymbol{V}^c_{\widehat{\boldsymbol{\mathcal{R}}},F}) \\ &- \int_F w_F \,\mathrm{div_F}(\boldsymbol{V}_{\boldsymbol{\mathcal{P}},\boldsymbol{n}_F}) + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{w}_E \boldsymbol{V}_F \boldsymbol{n}_{FE}.\end{aligned} \tag{4.62}$$

We define the full operator $\underline{\nabla}^{k+1}_F : \underline{\boldsymbol{X}}^k_{\nabla,F} \to \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,F}$ by

$$\underline{\nabla}^{k+1}_F \underline{\boldsymbol{w}}_F = ((\nabla^{k+2}_E\, \boldsymbol{w}_E)_{E\in\mathcal{E}_F}, \nabla^{k+1}_F\, \underline{\boldsymbol{w}}_F). \tag{4.63}$$

We prove a first commutative property:

Lemma 4.15. *For all $F \in \mathcal{F}_h$ it holds:*

$$\nabla_F^{k+1}(\underline{\boldsymbol{I}}_{\nabla,F}^k \boldsymbol{w}) = \boldsymbol{\pi}_{\widetilde{\boldsymbol{\mathcal{P}}},F}^{k+1}(\nabla \boldsymbol{w}), \qquad \forall \boldsymbol{w} \in \boldsymbol{C}^1(\overline{F}). \tag{4.64}$$

Proof. For all $\boldsymbol{w} \in \boldsymbol{C}^1(\overline{F})$ and all $\boldsymbol{V}_F \in \widetilde{\boldsymbol{\mathcal{P}}}^{k+1}(F)$,

$$\begin{aligned}
\int_F \nabla_F^{k+1}(\underline{\boldsymbol{I}}_{\nabla,F}^k \boldsymbol{w}) : \boldsymbol{V}_F = &- \int_F \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}(\boldsymbol{w}_{t,F}) \cdot \nabla\cdot(\boldsymbol{V}_{\overline{\boldsymbol{\mathcal{R}}},F}^c) - \int_F \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{k}(\boldsymbol{w}_{t,F}) \cdot \nabla\cdot(\boldsymbol{V}_{\widehat{\boldsymbol{\mathcal{R}}},F}^c) \\
&- \int_F \pi_{\mathcal{P},F}^k(\boldsymbol{w}\cdot\boldsymbol{n}_F)\,\mathrm{div_F}(\boldsymbol{V}_{\boldsymbol{\mathcal{P}},\boldsymbol{n}_F}) + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1}(\boldsymbol{w}) \boldsymbol{V}_F \boldsymbol{n}_{FE} \\
= &- \int_F \boldsymbol{w}_{t,F} \cdot \nabla\cdot(\boldsymbol{V}_{\overline{\boldsymbol{\mathcal{R}}},F}^c + \boldsymbol{V}_{\widehat{\boldsymbol{\mathcal{R}}},F}^c) - \int_F (\boldsymbol{w}\cdot\boldsymbol{n}_F)\cdot \mathrm{div_F}(\boldsymbol{V}_{\boldsymbol{\mathcal{P}},\boldsymbol{n}_F})) \\
&+ \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{w} \boldsymbol{V}_F \boldsymbol{n}_{FE} \\
= &- \int_F \boldsymbol{w} \cdot \nabla\cdot(\boldsymbol{V}_F) + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{w} \boldsymbol{V}_F \boldsymbol{n}_{FE} \\
= &\int_F \nabla \boldsymbol{w} : \boldsymbol{V}_F.
\end{aligned}$$

We used Lemma 4.8 and the definition (4.18) to remove the first two projections ($\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}$ and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{k}$) and integration by parts to conclude. □

We define the trace operator $\gamma_{\nabla,F}^{k+2} : \underline{\boldsymbol{X}}_{\nabla,F}^k \to \mathcal{P}^{k+2}(F;\mathbb{R}^3)$ by the relation: $\forall \boldsymbol{V}_F \in (\boldsymbol{\mathcal{R}}^{c,k+3}(F)^\intercal)^3$, $\forall \underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}_{\nabla,F}^k$,

$$\int_F \gamma_{\nabla,F}^{k+2}(\underline{\boldsymbol{w}}_F) \cdot \nabla\cdot \boldsymbol{V}_F = -\int_F \nabla_F^{k+1} \underline{\boldsymbol{w}}_F : \boldsymbol{V}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{w}_E \boldsymbol{V}_F \boldsymbol{n}_{FE}. \tag{4.65}$$

The isomorphism (4.11) ensures the well-posedness.

Remark 4.16. The relation (4.65) also holds for all $\boldsymbol{V}_F \in (\boldsymbol{\mathcal{P}}^{k+1}(F)^\intercal)^3$. Indeed if $\boldsymbol{V}_F$ belongs to $(\boldsymbol{\mathcal{R}}^{k+1}(F)^\intercal)^3$ then $\nabla\cdot \boldsymbol{V}_F = 0$ and the left-hand side of (4.65) vanishes, and since $(\boldsymbol{\mathcal{R}}^{k+1}(F)^\intercal)^3 \subset (\boldsymbol{\mathcal{P}}^{k+1}(F)^\intercal)^3 \approx \widetilde{\boldsymbol{\mathcal{P}}}^{k+1}(F)$ we can apply (4.62) to show that the right-hand side is also zero. Hence, the relation holds for all $(\boldsymbol{\mathcal{R}}^{k+1}(F)^\intercal)^3 \oplus (\boldsymbol{\mathcal{R}}^{c,k+3}(F)^\intercal)^3 \supset (\boldsymbol{\mathcal{P}}^{k+1}(F)^\intercal)^3$.

Lemma 4.17 (Consistency properties)**.** *For all $F \in \mathcal{F}_h$ the following relations hold:*

$$\gamma_{\nabla,F}^{k+2}(\underline{\boldsymbol{I}}_{\nabla,F}^k \boldsymbol{w}) = \boldsymbol{w}, \qquad \forall \boldsymbol{w} \in \mathcal{P}^{k+2}(F;\mathbb{R}^3), \tag{4.66}$$

$$\left.\begin{aligned}
\pi_{\mathcal{P},F}^k(\gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{w}}_F \cdot \boldsymbol{n}_F) &= w_F, \\
\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}(\gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{w}}_F) &= \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}^c, \\
\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{k}(\gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{w}}_F) &= \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F},
\end{aligned}\right. \qquad \forall \underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}_{\nabla,F}^k. \tag{4.67}$$

Proof. Proof of (4.67). Let $\underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ and $\boldsymbol{v}_F \in \boldsymbol{\mathcal{R}}^{c,k+1}(F)$. Remark 4.16 and the definition (4.62) of ∇_F^{k+1} allow us to write:

$$\begin{aligned}\int_F \gamma^{k+2}_{\nabla,F}(\underline{\boldsymbol{w}}_F)\cdot\boldsymbol{n}_F\ \mathrm{div}_\mathrm{F}(\boldsymbol{v}_F) &= \int_F \gamma^{k+2}_{\nabla,F}(\underline{\boldsymbol{w}}_F)\cdot\nabla\cdot(\boldsymbol{n}_F\otimes\boldsymbol{v}_F)\\ &= -\int_F \nabla_F^{k+1}\,\underline{\boldsymbol{w}}_F:(\boldsymbol{n}_F\otimes\boldsymbol{v}_F)\\ &\quad+\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E(\boldsymbol{w}_E\cdot\boldsymbol{n}_F)(\boldsymbol{v}_F\cdot\boldsymbol{n}_{FE})\\ &= \int_F w_F\,\mathrm{div}_\mathrm{F}(\boldsymbol{v}_F).\end{aligned}$$

Since this holds for all $\boldsymbol{v}_F \in \boldsymbol{\mathcal{R}}^{c,k+1}(F)$, from isomorphism (4.11) we have:

$$\pi^k_{\mathcal{P},F}(\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\cdot\boldsymbol{n}_F) = \pi^k_{\mathcal{P},F}(w_F) = w_F.$$

The other two equations are proven in the same fashion.
Proof of (4.66). Let $\boldsymbol{w}\in\mathcal{P}^{k+2}(F;\mathbb{R}^3)$ and $\boldsymbol{V}_F\in(\boldsymbol{\mathcal{R}}^{c,k+3}(F)^\intercal)^3$, it holds

$$\begin{aligned}\int_F \gamma^{k+2}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{w})\cdot\nabla\cdot\boldsymbol{V}_F &= -\int_F\nabla_F^{k+1}\,\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{w}:\boldsymbol{V}_F+\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E(\boldsymbol{w}_E)\boldsymbol{V}_F\boldsymbol{n}_{FE}\\ &= -\int_F\nabla\,\boldsymbol{w}:\boldsymbol{V}_F+\sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E\boldsymbol{w}\boldsymbol{V}_F\boldsymbol{n}_{FE}\\ &= \int_F\boldsymbol{w}\cdot\nabla\cdot\boldsymbol{V}_F.\end{aligned}$$

We used Lemma 4.15 to write $\nabla_F^{k+1}\,\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{w} = \boldsymbol{\pi}^{k+1}_{\widetilde{\boldsymbol{\mathcal{P}}},F}(\nabla\,\boldsymbol{w}) = \nabla\,\boldsymbol{w}$ since $\nabla\,\boldsymbol{w}\in(\mathcal{P}^{k+1}(F;\mathbb{R}^3)^\intercal)^3$, and $\boldsymbol{w}_E = \boldsymbol{w}$ since $\boldsymbol{w}\in(\mathcal{P}^{k+2})^3\subset(\mathcal{P}^{k+3})^3$ is continuous to remove the projections. □

For all $T\in\mathcal{T}_h$ we define the operator $\nabla_T^{k+1}:\underline{\boldsymbol{X}}^k_{\nabla,T}\to\overline{\boldsymbol{\mathcal{RT}}}^{k+1}(T)$ such that $\forall\underline{\boldsymbol{w}}_T = ((\underline{\boldsymbol{w}}_F)_{F\in\mathcal{F}_T},\boldsymbol{w}_{\boldsymbol{\mathcal{G}},T},\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T})\in\underline{\boldsymbol{X}}^k_{\nabla,T}$, $\forall\boldsymbol{V}_T = \boldsymbol{V}^c_{\overline{\boldsymbol{\mathcal{R}}},T}+\boldsymbol{V}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}+\boldsymbol{V}_{\boldsymbol{\mathcal{R}},T}\in\overline{\boldsymbol{\mathcal{RT}}}^{k+1}(T)$,

$$\begin{aligned}\int_T\nabla_T^{k+1}(\underline{\boldsymbol{w}}_T):\boldsymbol{V}_T = -\int_T\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T}\cdot\nabla\cdot(\boldsymbol{V}^c_{\overline{\boldsymbol{\mathcal{R}}},T})-\int_T\boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}\cdot\nabla\cdot(\boldsymbol{V}^c_{\widehat{\boldsymbol{\mathcal{R}}},T})\\ +\sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\boldsymbol{V}_T\boldsymbol{n}_F.\end{aligned}\tag{4.68}$$

We also define the potential reconstruction operator $P^{k+1}_{\nabla,T} : \underline{\boldsymbol{X}}^k_{\nabla,T} \to \boldsymbol{\mathcal{P}}^{k+1}(T)$ by the relation: $\forall \boldsymbol{V}_T \in (\boldsymbol{\mathcal{R}}^{c,k+2}(T)^\intercal)^3$, $\forall \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$,

$$\int_T P^{k+1}_{\nabla,T}(\underline{\boldsymbol{w}}_T) \cdot \nabla\cdot \boldsymbol{V}_T = -\int_T \nabla^{k+1}_T \underline{\boldsymbol{w}}_T : \boldsymbol{V}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \boldsymbol{V}_T \boldsymbol{n}_F. \quad (4.69)$$

The global operator $\underline{\nabla}^{k+1}_T : \underline{\boldsymbol{X}}^k_{\nabla,T} \to \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,T}$ is defined for all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ by

$$\underline{\nabla}^{k+1}_T \underline{\boldsymbol{w}}_T = ((\underline{\nabla}^{k+1}_F \underline{\boldsymbol{w}}_F)_{F\in\mathcal{F}_T}, \nabla^{k+1}_T \underline{\boldsymbol{w}}_T). \quad (4.70)$$

Remark 4.18. Since $\nabla\cdot(\boldsymbol{\mathcal{R}}^k(T)^\intercal)^3 = 0$ by Remark 4.6 and (4.67) we see that $\forall q \in \mathcal{P}^k(T)$,

$$\int_T \nabla^{k+1}_T(\underline{\boldsymbol{w}}_T) :(q\,\mathbf{I}_{3,3}) = \int_T \boldsymbol{w}_{\boldsymbol{\mathcal{G}},T} \cdot \mathbf{grad}\, q + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F q\, w_F.$$

Remark 4.19. With the same argument as in Remark 4.16, (4.69) is valid for all $\boldsymbol{V}_T \in (\boldsymbol{\mathcal{P}}^{k+1}(T)^\intercal)^3$.

Lemma 4.20. *For all $T \in \mathcal{T}_h$ it holds:*

$$\nabla^{k+1}_T(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) = \boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{RT}}},T}(\nabla\,\boldsymbol{w}), \qquad \forall \boldsymbol{w} \in \boldsymbol{C}^1(\overline{T}), \quad (4.71)$$

$$P^{k+1}_{\nabla,T}(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) = \boldsymbol{w}, \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{P}}^{k+1}(T), \quad (4.72)$$

$$\begin{aligned} \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},T}(P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T) &= \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T}, \\ \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},T}(P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T) &= \boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}, \end{aligned} \qquad \forall \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}. \quad (4.73)$$

Proof. Proof of (4.71). Since $\overline{\boldsymbol{\mathcal{R}}}^{c,k+1} \subset (\boldsymbol{\mathcal{R}}^{c,k+1}(T)^\intercal)^3$, from the definition (4.19) and from (4.14), it holds $\forall F \in \mathcal{F}_T$, $\forall \boldsymbol{V}_T \in \overline{\boldsymbol{\mathcal{RT}}}^{k+1}(T)$, $\boldsymbol{V}_T\,\boldsymbol{n}_F \in \mathcal{P}^k(F;\mathbb{R}^3)$. This allows using (4.67) to remove the projections on the boundary terms in the definition (4.68), and proceed as in the proof of Lemma 4.15. The proofs of (4.72) and (4.73) are the same as the proofs of (4.66) and (4.67). □

Divergence.

Finally, we define the discrete divergence operator, for all $T \in \mathcal{T}_h$ by:

$$D^k_T := \operatorname{Tr} \nabla^{k+1}_T \in \mathcal{P}^k(T).$$

As in the continuous case the divergence is the trace of the gradient, but we can also define it by a formula mimicking the integration by parts. By Remark 4.18,

$\forall \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$, D^k_T is such that $\forall q_T \in \mathcal{P}^k(T)$,

$$\begin{aligned}\int_T D^k_T \underline{\boldsymbol{w}}_T q_T &= \int_T \mathrm{Tr}(\nabla^{k+1}_T \underline{\boldsymbol{w}}_T) q_T \\ &= \int_T \nabla^{k+1}_T \underline{\boldsymbol{w}}_T : (q_T \mathbf{I}_{3,3}) \\ &= -\int_T \boldsymbol{w}_{\boldsymbol{\mathcal{G}},T} \cdot \mathbf{grad}\, q_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F (\gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \cdot \boldsymbol{n}_F) q_T \\ &= -\int_T \boldsymbol{w}_{\boldsymbol{\mathcal{G}},T} \cdot \mathbf{grad}\, q_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F w_F q_T.\end{aligned} \tag{4.74}$$

We get the same definition as the one of the discrete de Rham complex of [67].

4.3.4 Discrete L^2-product.

We build scalar products on the discrete spaces. They are made of the sum of the L^2 scalar product on each cell and of a stabilization term taking the lower dimensional objects (edges, vertices and faces) into account. First we define them locally for all $T \in \mathcal{T}_h$: For all $\underline{q}_T, \underline{r}_T \in \underline{X}^k_{\mathbf{grad},T}$ we set

$$\left(\underline{q}_T, \underline{r}_T\right)_{\mathbf{grad},T} = \int_T P^{k+1}_{\mathbf{grad},T}\underline{q}_T \cdot P^{k+1}_{\mathbf{grad},T}\underline{r}_T + \mathrm{s}_{\mathbf{grad},T}\left(\underline{q}_T, \underline{r}_T\right), \tag{4.75}$$

$$\begin{aligned}\mathrm{s}_{\mathbf{grad},T}\left(\underline{q}_T, \underline{r}_T\right) = \sum_{F\in\mathcal{F}_T} h_F \Bigg[\int_F &(P^{k+1}_{\mathbf{grad},T}\underline{q}_T - \gamma^{k+1}_{\mathbf{grad},F}\underline{q}_F) \cdot (P^{k+1}_{\mathbf{grad},T}\underline{r}_T - \gamma^{k+1}_{\mathbf{grad},F}\underline{r}_F) \\ &+ h_F^2 \int_F (\pi^{k-1}_{\mathcal{P},F} \partial_{\boldsymbol{n}_F} P^{k+1}_{\mathbf{grad},T}\underline{q}_T - G_{q,F}) \cdot (\pi^{k-1}_{\mathcal{P},F} \partial_{\boldsymbol{n}_F} P^{k+1}_{\mathbf{grad},T}\underline{r}_T - G_{r,F})\Bigg] \\ &+ \sum_{E\in\mathcal{E}_T} h_E^2 \Bigg[\int_E (P^{k+1}_{\mathbf{grad},T}\underline{q}_T - q_E) \cdot (P^{k+1}_{\mathbf{grad},T}\underline{r}_T - r_E) \\ &\quad + h_E^2 \int_E (\boldsymbol{t}_E \times (\mathbf{grad}\, P^{k+1}_{\mathbf{grad},T}\underline{q}_T \times \boldsymbol{t}_E) - \boldsymbol{G}_{q,E}) \\ &\quad\quad \cdot (\boldsymbol{t}_E \times (\mathbf{grad}\, P^{k+1}_{\mathbf{grad},T}\underline{r}_T \times \boldsymbol{t}_E) - \boldsymbol{G}_{r,E})\Bigg] \\ &+ \sum_{V\in\mathcal{V}_T} h_T^5 (\mathbf{grad}\, P^{k+1}_{\mathbf{grad},T}\underline{q}_T(\boldsymbol{x}_V) - \boldsymbol{G}_{q,V}) \\ &\quad\quad \cdot (\mathbf{grad}\, P^{k+1}_{\mathbf{grad},T}\underline{r}_T(\boldsymbol{x}_V) - \boldsymbol{G}_{r,V}).\end{aligned} \tag{4.76}$$

For all $\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$ we set

$$(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T)_{\mathbf{curl},T} = \int_T \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T \cdot \boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T + \mathrm{s}_{\mathbf{curl},T}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T), \tag{4.77}$$

$$
\begin{aligned}
\mathrm{s}_{\mathbf{curl},T}\left(\underline{\boldsymbol{v}}_T,\underline{\boldsymbol{w}}_T\right) = \sum_{F\in\mathcal{F}_T} h_F \Bigg[& \int_F (\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_{T_{t,F}} - \gamma^k_{t,\mathrm{rot},F}\underline{\boldsymbol{v}}_F)\cdot(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_{T_{t,F}} - \gamma^k_{t,\mathrm{rot},F}\underline{\boldsymbol{w}}_F) \\
& \int_F (\pi^{k-1}_{\mathcal{P},F}\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T\cdot\boldsymbol{n}_F - v_F)\cdot(\pi^{k-1}_{\mathcal{P},F}\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T\cdot\boldsymbol{n}_F - w_F) \\
& + h_F^2\int_F (\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{n}_F\times(\nabla\,\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T)) - \boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}) \\
& \qquad\cdot(\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{n}_F\times(\nabla\,\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T)) - \boldsymbol{R}_{w,\boldsymbol{\mathcal{G}},F}) \\
& + h_F^2\int_F (\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{n}_F\times(\nabla\,\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T)) - \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F}) \\
& \qquad\cdot(\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\boldsymbol{n}_F\times(\nabla\,\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T)) - \boldsymbol{R}^c_{w,\boldsymbol{\mathcal{G}},F})\Bigg] \\
+ \sum_{E\in\mathcal{E}_T} h_E^2 \Bigg[& \int_E (\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T - v_E)\cdot(\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T - w_E) \\
& + h_E^2\int_E (\mathbf{c}_E\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T - \boldsymbol{R}_{v,E})\cdot(\mathbf{c}_E\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T - \boldsymbol{R}_{w,E})\Bigg] \\
+ \sum_{V\in\mathcal{V}_T} & h_T^5(\mathbf{curl}\,\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{v}}_T(\boldsymbol{x}_V) - \boldsymbol{R}_{v,V})\cdot(\mathbf{curl}\,\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T(\boldsymbol{x}_V) - \boldsymbol{R}_{w,V}),
\end{aligned}
\tag{4.78}
$$

where $\mathbf{c}_E\boldsymbol{v} := (\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{t}_E)\,\boldsymbol{t}_E + \mathbf{grad}(\boldsymbol{v}\cdot\boldsymbol{t}_E)\times\boldsymbol{t}_E$. For all $\underline{\boldsymbol{v}}_T,\underline{\boldsymbol{w}}_T\in\underline{\boldsymbol{X}}^k_{\nabla,T}$ we set

$$
(\underline{\boldsymbol{v}}_T,\underline{\boldsymbol{w}}_T)_{\nabla,T} = \int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{v}}_T\cdot P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T + \mathrm{s}_{\nabla,T}\left(\underline{\boldsymbol{v}}_T,\underline{\boldsymbol{w}}_T\right), \tag{4.79}
$$

$$
\begin{aligned}
\mathrm{s}_{\nabla,T}\left(\underline{\boldsymbol{v}}_T,\underline{\boldsymbol{w}}_T\right) = {} & \sum_{F\in\mathcal{F}_T} h_F\int_F (P^{k+1}_{\nabla,T}\underline{\boldsymbol{v}}_T - \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{v}}_F)\cdot(P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T - \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{v}}_F) \\
& + \sum_{E\in\mathcal{E}_F} h_E^2\int_E (P^{k+1}_{\nabla,T}\underline{\boldsymbol{v}}_T - \boldsymbol{v}_E)\cdot(P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T - \boldsymbol{w}_E).
\end{aligned}
\tag{4.80}
$$

For all $\underline{\boldsymbol{V}}_T,\underline{\boldsymbol{W}}_T\in\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,T}$ we set

$$
(\underline{\boldsymbol{V}}_T,\underline{\boldsymbol{W}}_T)_{\boldsymbol{L}^2,T} = \int_T \boldsymbol{V}_T:\boldsymbol{W}_T + \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{V}}_T,\underline{\boldsymbol{W}}_T\right), \tag{4.81}
$$

$$
\begin{aligned}
\mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{V}}_T,\underline{\boldsymbol{W}}_T\right) = {} & \sum_{F\in\mathcal{F}_T} h_F\int_F (\boldsymbol{V}_{\otimes t,F} - \boldsymbol{V}_F):(\boldsymbol{W}_{\otimes t,F} - \boldsymbol{W}_F) \\
& + \sum_{E\in\mathcal{E}_F} h_E^2\int_E (\boldsymbol{V}_T\,\boldsymbol{t}_E - \boldsymbol{V}_E)\cdot(\boldsymbol{W}_T\,\boldsymbol{t}_E - \boldsymbol{W}_E),
\end{aligned}
\tag{4.82}
$$

where $\boldsymbol{V}_{\otimes t,F} := \boldsymbol{V}_T - (\boldsymbol{V}_T\ \boldsymbol{n}_F)\otimes\boldsymbol{n}_F$. Global scalar products are then merely the sum of local scalar product over every element $T\in\mathcal{T}_h$. For all $\underline{q}_T\in\underline{X}^k_{\mathbf{grad},T}$,

$\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$, $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ and $\underline{\boldsymbol{W}}_T \in \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,T}$ the norm induced by this scalar product is denoted by:

$$\|\underline{q}_T\|_{\mathbf{grad},T} = \left(\underline{q}_T, \underline{q}_T\right)^{1/2}_{\mathbf{grad},T}, \quad \|\underline{\boldsymbol{v}}_T\|_{\mathbf{curl},T} = (\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T)^{1/2}_{\mathbf{curl},T},$$
$$\|\underline{\boldsymbol{v}}_T\|_{\nabla,T} = (\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T)^{1/2}_{\nabla,T}, \quad \|\underline{\boldsymbol{W}}_T\|_{\boldsymbol{L}^2,T} = (\underline{\boldsymbol{W}}_T, \underline{\boldsymbol{W}}_T)^{1/2}_{\boldsymbol{L}^2,T}.$$

We also define norms built from the sum over the objects of every dimension. For all $\underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}$ we define

$$\begin{aligned}\left\|\left|\underline{q}_T\right|\right\|^2_{\mathbf{grad},T} = \; & \|q_T\|^2_T + \sum_{F\in\mathcal{F}_T} h_F \left(\|q_F\|^2_F + h_F^2 \|G_{q,F}\|^2_F\right. \\ & \left. + \sum_{E\in EF} h_E \left[\|q_E\|^2_E + h_E^2 \|\boldsymbol{G}_{q,E}\|^2_E + \sum_{V\in\mathcal{V}_E} h_E^3 |\boldsymbol{G}_{q,V}|^2\right]\right).\end{aligned} \tag{4.83}$$

For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$ we define

$$\begin{aligned}|||\underline{\boldsymbol{v}}_T|||^2_{\mathbf{curl},T} = \; & \|\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}\|^2_T + \left\|\boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},T}\right\|^2_T + \sum_{F\in\mathcal{F}_T} h_F \left(\|\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}\|^2_F + \left\|\boldsymbol{v}^c_{\boldsymbol{\mathcal{R}},F}\right\|^2_F + \|v_F\|^2_F\right. \\ & \left. + h_F^2 \|\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}\|^2_F + h_F^2 \left\|\boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F}\right\|^2_F + \sum_{E\in\mathcal{E}_F} h_E \left[\|\boldsymbol{v}_E\|^2_E + h_E^2 \|\boldsymbol{R}_{v,E}\|^2_E + \sum_{V\in\mathcal{V}_E} h_E^3 |\boldsymbol{R}_{v,V}|^2\right]\right).\end{aligned} \tag{4.84}$$

For all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ we define

$$\begin{aligned}|||\underline{\boldsymbol{w}}_T|||^2_{\nabla,T} = \; & \|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}\|^2_T + \left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T}\right\|^2_T + \sum_{F\in\mathcal{F}_T} h_F \left(\|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}\|^2_F + \left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}\right\|^2_F + \|w_F\|^2_F\right. \\ & \left. + \sum_{E\in\mathcal{E}_F} h_E \|\boldsymbol{w}_E\|^2_E\right).\end{aligned} \tag{4.85}$$

For all $\underline{\boldsymbol{W}}_T \in \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,T}$ we define

$$|||\underline{\boldsymbol{W}}_T|||^2_{\boldsymbol{L}^2,T} = \|\boldsymbol{W}_T\|^2_T + \sum_{F\in\mathcal{F}_T} h_F \left(\|\boldsymbol{W}_F\|^2_F + \sum_{E\in\mathcal{E}_F} h_E \|\boldsymbol{W}_E\|^2_E\right). \tag{4.86}$$

We show the equivalence between the norm induced by the discrete L^2-products (4.75), (4.77), (4.79) and (4.81) and the component-wise norm (4.83), (4.84), (4.85) and (4.86)in Lemma 4.25.

We define the global norms over Ω as the sum of the local norms over every cell $T \in \mathcal{T}_h$, i.e. $|||\underline{\boldsymbol{v}}_h|||^2_{\nabla,h} = \sum_{T\in\mathcal{T}_h} |||\underline{\boldsymbol{v}}_T|||^2_{\nabla,T}$.

4.3.5 Results on discrete L^2-products.

We show some results to justify the choice of discrete norms.

Lemma 4.21. *For all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ it holds:*

$$\left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F \lesssim h_F^{-1} |||\underline{\boldsymbol{w}}_F|||_{\nabla,F}.$$

Proof. Let $F \in \mathcal{F}_h$ and $\underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$. The proof hinges on the bound of the divergence $|||\nabla \cdot||| \lesssim h^{-1}$, the definition (4.62) with $\boldsymbol{V}_F = \nabla^{k+1}_F \underline{\boldsymbol{w}}_F$ and the boundedness of the topological decomposition (4.20) to show that:

$$\begin{aligned}
\int_F \nabla^{k+1}_F \underline{\boldsymbol{w}}_F : \nabla^{k+1}_F \underline{\boldsymbol{w}}_F \lesssim{}& \left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}\right\|_F h^{-1} \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F + \left\|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}\right\|_F h^{-1} \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F \\
&+ \left\|w_F\right\|_F h^{-1} \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F + \sum_{E \in \mathcal{E}_F} \left\|\boldsymbol{w}_E\right\|_E \ h^{-\frac{1}{2}} \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F \\
\lesssim{}& h^{-1} \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F \Bigg(\left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}\right\|_F + \left\|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}\right\|_F + \left\|w_F\right\|_F \\
&+ \sum_{E \in \mathcal{E}_F} h^{\frac{1}{2}} \left\|\boldsymbol{w}_E\right\|_E \Bigg) \\
\lesssim{}& h^{-1} \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F |||\underline{\boldsymbol{w}}_F|||_{\nabla,F}.
\end{aligned}$$

□

Lemma 4.22 (Boundedness of the local trace)**.** *For all $F \in \mathcal{F}_h$ and all $\underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$ it holds:*

$$\left\|\gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F\right\|_F \lesssim \left\|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}\right\|_F + \left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}\right\|_F + \left\|w_F\right\|_F + \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \left\|\boldsymbol{w}_E\right\|_E .$$

Proof. For any $\underline{\boldsymbol{w}}_F \in \underline{\boldsymbol{X}}^k_{\nabla,F}$, let $\boldsymbol{V}_F \in (\boldsymbol{\mathcal{R}}^{c,k+3}(F)^\intercal)^3$ be such that $\nabla \cdot \boldsymbol{V}_F = \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F$. From the estimate on the isomorphism (4.11) it holds $\left\|\boldsymbol{V}_F\right\|_F \lesssim h_F \left\|\gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F\right\|_F$. Then from (4.65) we write:

$$\begin{aligned}
\int_F \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \cdot \nabla \cdot \boldsymbol{V}_F ={}& -\int_F \nabla^{k+1}_F \underline{\boldsymbol{w}}_F : \boldsymbol{V}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{w}_E \boldsymbol{V}_F \boldsymbol{n}_{FE} \\
\lesssim{}& \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F \left\|\boldsymbol{V}_F\right\|_F + \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \left\|\boldsymbol{w}_E\right\|_E h_E^{-1} \left\|\boldsymbol{V}_F\right\|_F \\
\lesssim{}& \left(h_F \left\|\nabla^{k+1}_F \underline{\boldsymbol{w}}_F\right\|_F + \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \left\|\boldsymbol{w}_E\right\|_E \right) \left\|\gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F\right\|_F .
\end{aligned}$$

We used the discrete trace inequality (Lemma 4.3) on the boundary term, and we conclude with Lemma 4.21. □

Lemma 4.23 (Discrete inverse inequality)**.** *For all $T \in \mathcal{T}_h$ and all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ it holds:*

$$\left\|\nabla_T^{k+1}\, \underline{\boldsymbol{w}}_T\right\| \lesssim h^{-1} |||\underline{\boldsymbol{w}}_T|||_{\nabla,T}.$$

Proof. The proof is similar to the proof of Lemma 4.21, using the boundedness of the local trace (Lemma 4.22) to conclude. □

Lemma 4.24 (Boundedness of the local potential)**.** *For all $T \in \mathcal{T}_h$ and all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ it holds:*

$$\left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\| \lesssim |||\underline{\boldsymbol{w}}_T|||_{\nabla,T}. \tag{4.87}$$

Proof. The proof is similar to the proof of Lemma 4.22. □

The boundedness of the local potentials $P^{k+1}_{\mathbf{grad},T}$ and $\boldsymbol{P}^k_{\mathbf{curl},T}$ follows directly from the corresponding results of [67] because we use the same potentials with stronger component-wise norms.

Lemma 4.25 (Equivalence of local norms)**.** *It holds, for all $T \in \mathcal{T}_h$*

$$\|\underline{q}_T\|_{\mathbf{grad},T} \approx \left|\left|\left|\underline{q}_T\right|\right|\right|_{\mathbf{grad},T}, \quad \forall \underline{q}_T \in \underline{X}^k_{\mathbf{grad},T}, \tag{4.88}$$

$$\|\underline{\boldsymbol{v}}_T\|_{\mathbf{curl},T} \approx |||\underline{\boldsymbol{v}}_T|||_{\mathbf{curl},T}, \quad \forall \underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},T}, \tag{4.89}$$

$$\|\underline{\boldsymbol{w}}_T\|_{\nabla,T} \approx |||\underline{\boldsymbol{w}}_T|||_{\nabla,T}, \quad \forall \underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}, \tag{4.90}$$

$$\|\underline{\boldsymbol{W}}_T\|_{\boldsymbol{L}^2,T} \approx |||\underline{\boldsymbol{W}}_T|||_{\boldsymbol{L}^2,T}, \quad \forall \underline{\boldsymbol{W}}_T \in \underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,T}. \tag{4.91}$$

Proof. We only prove (4.90), the other equivalences being established in a similar way. We apply the trace inequality to the definitions (4.79) and (4.80) to write:

$$\begin{aligned}
\|\underline{\boldsymbol{w}}_T\|^2_{\nabla,T} \leq{}& \left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_T + \sum_{F\in\mathcal{F}_T} h_F \left(\left\|\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\right\|^2_F + \left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_F\right) \\
&+ \sum_{E\in\mathcal{E}_T} h_E^2 \left(\|\boldsymbol{w}_E\|^2_E + \left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_E\right) \\
\lesssim{}& \left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_T + \sum_{F\in\mathcal{F}_T} h_F \left\|\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\right\|^2_F + \sum_{E\in\mathcal{E}_T} h_E^2 \left\|\boldsymbol{w}_E\right\|^2_E \\
\lesssim{}& |||\underline{\boldsymbol{w}}_T|||^2_{\nabla,T}.
\end{aligned}$$

We conclude with the boundedness of the local potential and trace (Lemma 4.24 and 4.22).

Conversely, to prove $|||\underline{\boldsymbol{w}}_T|||_{\nabla,T} \lesssim \|\underline{\boldsymbol{w}}_T\|_{\nabla,T}$ we begin to write

$$\begin{aligned}
\|\boldsymbol{w}_E\|^2_E &\lesssim \left\|\boldsymbol{w}_E - P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_E + h_E^{-2}\left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_T, \\
\left\|\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\right\|^2_F &\lesssim \left\|\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F - P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_F + h_F^{-1}\left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|^2_T.
\end{aligned}$$

Then we conclude with the consistency properties (4.73) and (4.67) which allow us to bound $\|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}\|_T$, $\left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},T}\right\|_T$, $\|\boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}\|_F$, $\left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}\right\|_F$ and $\|w_F\|_F$. For example $\left\|\boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F}\right\|_F = \left\|\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\right\|_F \leq \left\|\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\right\|_F$. □

4.4 Complex property.

In this section we study the following sequence:

$$\underline{X}^k_{\mathbf{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\nabla,h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h). \tag{4.92}$$

We will show in Theorem 4.27 that (4.92) is indeed a complex, but first we show that the interpolators commute with the differential operators.

Lemma 4.26 (Local commutation properties)**.** *It holds for all $T \in \mathcal{T}_h$,*

$$\underline{\boldsymbol{G}}^k_T(\underline{I}^k_{\mathbf{grad},T}q) = \underline{I}^k_{\mathbf{curl},T}(\mathbf{grad}\, q), \quad \forall q \in C^2(\overline{T}), \tag{4.93a}$$

$$\underline{\boldsymbol{C}}^k_T(\underline{I}^k_{\mathbf{curl},T}\boldsymbol{v}) = \underline{\boldsymbol{I}}^k_{\nabla,T}(\mathbf{curl}\,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{C}^1(\overline{T}), \tag{4.93b}$$

$$\underline{\nabla}^{k+1}_T(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) = \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}(\nabla\,\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \boldsymbol{C}^1(\overline{T}), \tag{4.93c}$$

$$D^k_T(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) = \pi^k_{\mathcal{P},T}(\operatorname{div}\boldsymbol{w}), \quad \forall \boldsymbol{w} \in \boldsymbol{C}^0(\overline{T}) \cap \boldsymbol{H}^1(T). \tag{4.93d}$$

Proof. Proof of (4.93a). We already have proved the relation on edges (4.39). Let $q \in C^2(\overline{T})$, we will show that the relation holds for $\boldsymbol{RG}^k_F$. The proofs for $\boldsymbol{G}^k_F$ and $\boldsymbol{G}^k_T$ are similar. For all $\boldsymbol{w}_F \in \boldsymbol{\mathcal{P}}^k(F)$,

$$\begin{aligned}
\int_F \boldsymbol{RG}^k_F(\underline{I}^k_{\mathbf{grad},F}q)\cdot\boldsymbol{w}_F &= -\int_F \pi^{k-1}_{\mathcal{P},F}(\partial_{\boldsymbol{n}_F}q)\operatorname{rot_F}\boldsymbol{w}_F - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \pi^k_{\mathcal{P},E}(\partial_{\boldsymbol{n}_F}q)\,\boldsymbol{w}_F\cdot\boldsymbol{t}_E\\
&= -\int_F \partial_{\boldsymbol{n}_F}q\ \operatorname{rot_F}\boldsymbol{w}_F - \sum_{E\in\mathcal{E}_F}\omega_{FE}\int_E \partial_{\boldsymbol{n}_F}q\,\boldsymbol{w}_F\cdot\boldsymbol{t}_E\\
&= -\int_F \mathbf{rot_F}(\partial_{\boldsymbol{n}_F}q)\cdot\boldsymbol{w}_F.
\end{aligned} \tag{4.94}$$

We conclude applying (4.94) for $\boldsymbol{w}_F \in \boldsymbol{\mathcal{G}}^k(F)$ and $\boldsymbol{w}_F \in \boldsymbol{\mathcal{G}}^{c,k}(F)$ since $\boldsymbol{n}_F \times (\nabla(\mathbf{grad}\,q)\cdot\boldsymbol{n}_F) = -\,\mathbf{rot_F}(\partial_{\boldsymbol{n}_F}q)$.

Proof of (4.93b). Let $\boldsymbol{v} \in C^1(\overline{T})$ and $\underline{\boldsymbol{v}}_T = \underline{I}^k_{\mathbf{curl},T}\boldsymbol{v}$. We will show the property for $\boldsymbol{C}^k_E$ and $\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}C^k_F + \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F}$, the other components are easier and similar. The same proof as (4.39) shows that $\boldsymbol{v}_E{}' = \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\partial_{\boldsymbol{t}_E}\boldsymbol{v})$. Let us choose an arbitrary basis $(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2)$ such that $\boldsymbol{t}_E = (1,0,0)$ and write $\boldsymbol{v} = (v_0, v_1, v_2)$. In this basis we have

$$\begin{aligned}
\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{C}^k_E \underline{\boldsymbol{v}}_E) &= \boldsymbol{R}_{v,E} - \boldsymbol{v}_E{}' \times \boldsymbol{t}_E \\
&= \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}\left[(\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{t}_E)\boldsymbol{t}_E + \mathbf{grad}(\boldsymbol{v}\cdot\boldsymbol{t}_E)\times\boldsymbol{t}_E\right] + \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\partial_0\boldsymbol{v})\times\boldsymbol{t}_E \\
&= \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}\left[\begin{pmatrix}\partial_1 v_2 - \partial_2 v_1\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \partial_2 v_0 - \partial_0 v_2\\ \partial_0 v_1 - \partial_1 v_0\end{pmatrix}\right] \\
&= \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\mathbf{curl}\,\boldsymbol{v}).
\end{aligned}$$

Now let us take another basis such that $\boldsymbol{n}_F = (1,0,0)$. By the same argument we have $C^k_F(\underline{I}^k_{\mathbf{curl},F}\boldsymbol{v}) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},F}(\mathbf{rot_F}\,v_0)$ so

$$\begin{aligned}
\boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(C^k_F\underline{\boldsymbol{v}}_F) + \boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F} &= \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}\left[\begin{pmatrix}0\\ \partial_2 v_0\\ -\partial_1 v_0\end{pmatrix}\right] + \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}\left[\begin{pmatrix}0\\ -\partial_0 v_2\\ \partial_0 v_1\end{pmatrix}\right] \\
&= \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}(\mathbf{curl}\,\boldsymbol{v}_{t,F}).
\end{aligned}$$

Proof of (4.93c). This is an immediate consequence of (4.61), (4.64) and (4.71).
Proof of (4.93d). Let $\boldsymbol{w} \in \boldsymbol{C}^0(\overline{T}) \cap \boldsymbol{H}^1(T)$. For all $q_T \in \mathcal{P}^k(T)$, since $\mathbf{grad}\,q_T \in \boldsymbol{\mathcal{G}}^{k-1}(T)$, we have:

$$\begin{aligned}
\int_T D^k_T(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w})\,q_T &= -\int_T \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{w})\cdot\mathbf{grad}\,q_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \pi^k_{\mathcal{P},F}(\boldsymbol{w}\cdot\boldsymbol{n}_F)\,q_T \\
&= -\int_T \boldsymbol{w}\cdot\mathbf{grad}\,q_T + \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F (\boldsymbol{w}\cdot\boldsymbol{n}_F)\,q_T \\
&= \int_T \operatorname{div}\boldsymbol{w}\;q_T.
\end{aligned}$$

□

Theorem 4.27 (Complex property)**.** *It holds:*

$$\underline{I}^k_{\mathbf{grad},h}\mathbb{R} = \operatorname{Ker}\underline{\boldsymbol{G}}^k_h, \tag{4.95a}$$
$$\operatorname{Im}\underline{\boldsymbol{G}}^k_h \subset \operatorname{Ker}\underline{\boldsymbol{C}}^k_h, \tag{4.95b}$$
$$\operatorname{Im}\underline{\boldsymbol{C}}^k_h \subset \operatorname{Ker}\underline{D}^k_h, \tag{4.95c}$$
$$\operatorname{Im}\underline{D}^k_h = \mathcal{P}^k(\mathcal{T}_h). \tag{4.95d}$$

Proof. Proof of (4.95a). The inclusion $\underline{I}^k_{\mathbf{grad},h}\mathbb{R} \subset \operatorname{Ker}\underline{\boldsymbol{G}}^k_h$ follows directly from (4.93a). Conversely if $\underline{q}_h \in \underline{X}^k_{\mathbf{grad},h}$ is such that $\underline{\boldsymbol{G}}^k_h\underline{q}_h = 0$ then since $\underline{\boldsymbol{G}}^k_E\underline{q}_E = 0$, (4.32) implies $\boldsymbol{G}_{q,V} = 0$, $\forall V \in \mathcal{V}_h$, $\boldsymbol{G}_{q,E} = 0$ and $q_E{}' = 0$, $\forall E \in \mathcal{E}_h$. So q_E is constant on every edge, however q_E is also continuous and Ω has a single connected component. Thus, there is $C \in \mathbb{R}$ such that $\forall E \in \mathcal{E}_h$, $q_E \equiv C$. From (4.33) and (4.35) we have $\forall \boldsymbol{w}_F \in \boldsymbol{\mathcal{R}}^{c,k}(F)$,

$$0 = -\int_F q_F \operatorname{div_F} \boldsymbol{w}_F + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E C\boldsymbol{w}_F \cdot \boldsymbol{n}_{FE} = \int_F (C - q_F) \operatorname{div_F} \boldsymbol{w}_F.$$

Since $\operatorname{div_F} : \boldsymbol{\mathcal{R}}^{c,k}(F) \to \mathcal{P}^{k-1}(F)$ is onto we must have $q_F \equiv C$, $\forall F \in \mathcal{F}_h$. Likewise, since $\boldsymbol{G}_{q,E} = 0$, (4.34) and (4.35) give

$$-\int_F G_{q,F} \ \operatorname{rot_F} \boldsymbol{w}_F = 0, \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{P}}^k(F).$$

Once again we must have $G_{q,F} = 0$, $\forall F \in \mathcal{F}_h$. The same argument using (4.42) gives $\gamma^{k+1}_{\mathbf{grad},F}\underline{q}_F \equiv C$ and $q_T \equiv C$, $\forall T \in \mathcal{T}_h$. Thus $\operatorname{Ker} \underline{\boldsymbol{G}}^k_h \subset \underline{I}^k_{\mathbf{grad},h}\mathbb{R}$.

Proof of (4.95b). We already have $\operatorname{Im} \underline{\boldsymbol{G}}^k_F \subset \operatorname{Ker} \underline{\boldsymbol{C}}^k_F$ by the local complex property on faces Lemma 4.13. Let $\underline{q} \in \underline{X}^k_{\mathbf{grad},h}$. For any $T \in \mathcal{T}_h$, since we project on $\boldsymbol{\mathcal{G}}^{k-1}(T) \oplus \boldsymbol{\mathcal{G}}^{c,k}(T)$ in (4.58) it is enough to show $\int_T \boldsymbol{C}^k_T(\underline{\boldsymbol{G}}^k_T\underline{q}_T) \cdot \boldsymbol{r}_T = 0$, $\forall \boldsymbol{r}_T \in \boldsymbol{\mathcal{N}}^k(T)$. Starting from (4.57) we write

$$\begin{aligned}
\int_T \boldsymbol{C}^k_T(\underline{\boldsymbol{G}}^k_T\underline{q}_T) \cdot \boldsymbol{r}_T &= \int_T \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{R}},T}(\boldsymbol{G}^k_T\underline{q}_T) \cdot \mathbf{curl}\, \boldsymbol{r}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^k_{\mathrm{t,rot},F}(\underline{\boldsymbol{G}}^k_F\underline{q}_F) \cdot (\boldsymbol{r}_T \times \boldsymbol{n}_F) \\
&= \int_T (\boldsymbol{G}^k_T\underline{q}_T) \cdot \mathbf{curl}\, \boldsymbol{r}_T + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F (\boldsymbol{G}^k_F\underline{q}_F) \cdot (\boldsymbol{r}_T \times \boldsymbol{n}_F) \\
&= -\int_T q_T \ \operatorname{div}(\mathbf{curl}\, \boldsymbol{r}_T) + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^{k+1}_{\mathbf{grad},F}(\underline{q}_F)(\mathbf{curl}(\boldsymbol{r}_T) \cdot \boldsymbol{n}_F) \\
&\quad + \sum_{F\in\mathcal{F}_T} \omega_{TF} \left[-\int_F q_F \ \operatorname{div_F}(\boldsymbol{r}_T \times \boldsymbol{n}_F) \right. \\
&\qquad \left. + \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_E (\boldsymbol{r}_T \times \boldsymbol{n}_F) \cdot \boldsymbol{n}_{FE} \right] \\
&= \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F q_F \left[\mathbf{curl}(\boldsymbol{r}_T) \cdot \boldsymbol{n}_F - \operatorname{div_F}(\boldsymbol{r}_T \times \boldsymbol{n}_F)\right] \\
&\quad - \sum_{F\in\mathcal{F}_T} \omega_{TF} \sum_{E\in\mathcal{E}_F} \omega_{FE} \int_E q_E \ \boldsymbol{r}_T \cdot \boldsymbol{t}_E.
\end{aligned}$$

We used $\mathbf{curl}\, \boldsymbol{r}_T \in \boldsymbol{\mathcal{R}}^{k-1}$ (4.7) and $\boldsymbol{r}_T \times \boldsymbol{n}_F \subset \boldsymbol{\mathcal{RT}}^k(F)$ (4.14) along with (4.56) on the first line. Then we used (4.37) and (4.33) on the second line and

(4.43) on the last. We conclude by using $\mathbf{curl}(\boldsymbol{r}_T) \cdot \boldsymbol{n}_F = \mathrm{div}_F(\boldsymbol{r}_T \times \boldsymbol{n}_F)$ and $\sum_{F \in \mathcal{F}_T} \omega_{TF} \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E \; \boldsymbol{r}_T \cdot \boldsymbol{t}_E = 0$. The last sum is zero since each edge shares exactly two faces on $\partial \overline{T}$ with opposite orientation. Hence we are counting each term twice, with a different sign each time.
Proof of (4.95c). The same proof as [67, Theorem 20] works.
Proof of (4.95d). This follows directly from the commutation property (4.93d) and the surjectivity of the continuous divergence operator. See the proof of existence of Lemma 4.34 (which does not require any new assumption). □

The complex is exact if and only if the inclusions (4.95b) and (4.95c) are in fact equalities. The proof is a slight adaptation of [60, Section 4.3].

Theorem 4.28 (Exactness)**.** *Let b_1 (resp. b_2) be the first (resp. second) Betti number of Ω. The following equalities hold:*

$$\mathrm{Im}\, \underline{\boldsymbol{G}}_h^k = \mathrm{Ker}\, \underline{\boldsymbol{C}}_h^k, \quad \text{if } b_1 = 0, \tag{4.96a}$$

$$\mathrm{Im}\, \underline{\boldsymbol{C}}_h^k = \mathrm{Ker}\, \underline{D}_h^k, \quad \text{if } b_2 = 0. \tag{4.96b}$$

Proof. Proof of (4.96a). Let $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$ be such that $\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h = 0$. We want to find $\underline{q}_h \in \underline{X}_{\mathbf{grad},h}^k$ such that $\underline{\boldsymbol{G}}_h^k \underline{q} = \underline{\boldsymbol{v}}$. Starting from the proof of [67, Theorem 3.1], if $b_1 = 0$ (if Ω has no tunnel) and $\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h = 0$ then we can find $q \in \mathcal{P}_c^{k+1}(\mathcal{E}_h)$ such that $\forall E \in \mathcal{E}_h$, $q' = \pi_{\mathcal{P},E}^k(\boldsymbol{v}_E \cdot \boldsymbol{t}_E)$. Let $\boldsymbol{G}_{q,E} := \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k(\boldsymbol{t}_E \times (\boldsymbol{v}_E \times \boldsymbol{t}_E))$ and $\boldsymbol{G}_{q,V} = \boldsymbol{v}_E(\boldsymbol{x}_V)$. Since $\boldsymbol{C}_E^k \underline{\boldsymbol{v}}_E = 0 \implies \boldsymbol{R}_{v,E} = \boldsymbol{v}_E{}' \times \boldsymbol{t}_E$ and $\boldsymbol{R}_{v,V} = 0$ we have $\underline{\boldsymbol{v}}_E = \underline{\boldsymbol{G}}_E^k(\underline{q}_E)$. We must also have $\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F} = -\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^k(\boldsymbol{C}_F^k \underline{\boldsymbol{v}}_E)$ hence $\forall \boldsymbol{r}_F \in \boldsymbol{\mathcal{G}}^k(F)$:

$$\int_F \underline{\boldsymbol{C}}_F^k \underline{\boldsymbol{v}} \cdot \boldsymbol{r}_F = \int_F v_F \,\mathrm{rot}_F\, \boldsymbol{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F (\boldsymbol{v}_E \cdot \boldsymbol{n}_F)(\boldsymbol{r}_F \cdot \boldsymbol{t}_E),$$

$$-\int_F \boldsymbol{R}\boldsymbol{G}_F^k(\underline{q}) \cdot \boldsymbol{r}_F = \int_F G_{q,F} \,\mathrm{rot}_F\, \boldsymbol{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F (\boldsymbol{G}_{q,E} \cdot \boldsymbol{n}_F)(\boldsymbol{r}_F \cdot \boldsymbol{t}_E).$$

Thus we can take $G_{q,F} := v_F$. We construct q_F and q_T exactly as in [67, Theorem 3.2].
Proof of (4.96b). Let $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$ be such that $\underline{D}_h^k \underline{\boldsymbol{w}}_h = 0$. We want to find $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$ such that $\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h = \underline{\boldsymbol{w}}_h$. If $b_2 = 0$ (if Ω has no bubble) then [67, Theorem 3.2] provides $\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T} \in \boldsymbol{\mathcal{R}}^{k-1}(T)$, $\boldsymbol{v}_{\boldsymbol{\mathcal{R}},T}^c \in \boldsymbol{\mathcal{R}}^{c,k}(T)$, $\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F} \in \boldsymbol{\mathcal{R}}^{k-1}(F)$, $\boldsymbol{v}_{\boldsymbol{\mathcal{R}},F}^c \in \boldsymbol{\mathcal{R}}^{c,k}(F)$ and a tangential component (along $\boldsymbol{t}_E$) of $\pi_{\mathcal{P},E}^k(\boldsymbol{v}_E \cdot \boldsymbol{t}_E)$ such that $C_F^k \underline{\boldsymbol{v}}_F = w_F$ and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},T}^{k-1}(\boldsymbol{C}_T^k \underline{\boldsymbol{v}}_T) = \boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},T}^{c,k}(\boldsymbol{C}_T^k \underline{\boldsymbol{v}}_T) = \boldsymbol{w}_{\boldsymbol{\mathcal{G}},T}^c$. It remains to show that $\boldsymbol{C}_E^k$ and $\boldsymbol{C}_F^k$ are onto without using the above-mentioned degrees of freedom. Let $\boldsymbol{R}_{v,V} = \boldsymbol{w}_E(\boldsymbol{x}_V)$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^k(\boldsymbol{v}_E \times \boldsymbol{t}_E) = \boldsymbol{0}$, $\boldsymbol{R}_{v,E} := \boldsymbol{v}_E{}' \times \boldsymbol{t}_E + \boldsymbol{\pi}_{\boldsymbol{\mathcal{P}},E}^{k+1}(\boldsymbol{w}_E)$, $v_F := 0$, $\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F} := \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F} - \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^k(\boldsymbol{C}_F^k \underline{\boldsymbol{v}}_F)$ and $\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}^c := \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}^c - \boldsymbol{\pi}_{\boldsymbol{\mathcal{G}},F}^{c,k}(\boldsymbol{C}_F^k \underline{\boldsymbol{v}}_F)$ (this makes

sense since $\boldsymbol{C}_F^k$ does not depend on $\boldsymbol{R}_{v,\boldsymbol{\mathcal{G}},F}$ nor on $\boldsymbol{R}^c_{v,\boldsymbol{\mathcal{G}},F}$). It is easily checked that this choice gives $\underline{\boldsymbol{C}}_h^k\underline{\boldsymbol{v}}_h = \underline{\boldsymbol{w}}_h$. □

4.5 Consistency results.

The last things we need to show in order to efficiently use this complex are consistency results. First we show primal consistency results, controlling the error made when we use the interpolators. Then we show some Poincare type results useful to show stability, including a discrete counterpart to the right inverse for the divergence Lemma 4.34. Finally we show adjoint consistency results, which control the error made when we perform a discrete integration by parts.

Lemma 4.29 (Primal consistency)**.** *For all $T \in \mathcal{T}_h$ it holds:*

$$\left\| P^{k+1}_{\nabla,T}(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) - \boldsymbol{w} \right\| \lesssim h^{k+2} \left|\boldsymbol{w}\right|_{\boldsymbol{H}^{k+2}}, \quad \forall \boldsymbol{w} \in \boldsymbol{H}^{k+2}(T). \tag{4.97}$$

Proof. For all $T \in \mathcal{T}_h$, (4.72) shows that $P^{k+1}_{\nabla,T}\underline{\boldsymbol{I}}^k_{\nabla,T}$ is a projection on $\boldsymbol{\mathcal{P}}^{k+1}(T)$. Thus we just have to show that $\left\| P^{k+1}_{\nabla,T}(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) \right\| \lesssim \|\boldsymbol{w}\| + h \|\boldsymbol{w}\|_{\boldsymbol{H}^1} + h^2 \|\boldsymbol{w}\|_{\boldsymbol{H}^2}$ to conclude with the lemma on approximation properties of bounded projector [59, Lemma 1.43]. Starting from (4.87) we have

$$\begin{aligned}
\left\| P^{k+1}_{\nabla,T}(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}) \right\| &\lesssim \left|\!\left|\!\left| \underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w} \right|\!\right|\!\right|_{\nabla,T} \\
&\lesssim \left\| \boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},T}\boldsymbol{w} \right\|_T + \left\| \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},T}\boldsymbol{w} \right\|_T + \sum_{E\in\mathcal{E}_T} h_E \left\| \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}\boldsymbol{w} \right\|_E + \sum_{V\in\mathcal{V}_E} h_E^{\frac{3}{2}} |\boldsymbol{w}(\boldsymbol{x}_V)| \\
&\quad + \sum_{F\in\mathcal{F}_T} h_F^{\frac{1}{2}} \left(\left\| \pi^k_{\mathcal{P},F}(\boldsymbol{w}\cdot\boldsymbol{n}_F) \right\|_F + \left\| \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{G}},F}\boldsymbol{w}_{\boldsymbol{t},\boldsymbol{F}} \right\|_F + \left\| \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},F}\boldsymbol{w}_{\boldsymbol{t},\boldsymbol{F}} \right\|_F \right) \\
&\lesssim \|\boldsymbol{w}\|_T + h_T \left|\boldsymbol{w}\right|_{\boldsymbol{H}^1(T)} + h_T^2 \left|\boldsymbol{w}\right|_{\boldsymbol{H}^2(T)},
\end{aligned}$$

where we used the continuous trace inequality [59, Lemma 1.31] and the boundedness of L^2 projectors. □

The interpolator on $\underline{X}^k_{\mathbf{grad},T}$ and on $\underline{\boldsymbol{X}}^k_{\mathbf{curl},T}$ requires the value of a derivative of the function on vertices, which in turn needs a high regularity. This motivates the introduction of the notation $|\boldsymbol{v}|_{\boldsymbol{H}^{(k+i,j)}(T)}$. It is defined for all $T \in \mathcal{T}_h$ and $\boldsymbol{v} \in \boldsymbol{H}^{\max(k+i,j)}$ by:

$$|\boldsymbol{v}|_{\boldsymbol{H}^{(k+i,j)}(T)} = \sum_{l=k+i}^{\max(k+i,j)} h_T^{l-k-i} \left|\boldsymbol{v}\right|_{\boldsymbol{H}^l}. \tag{4.98}$$

We recover the notation used in [67] for $(i,j) = (1,2)$.

Lemma 4.30 (Stabilization forms consistency)**.** *For all $T \in \mathcal{T}_h$ it holds:*

$$s_{\mathbf{grad},T}\left(\underline{I}^k_{\mathbf{grad},T}q, \underline{I}^k_{\mathbf{grad},T}q\right)^{1/2} \lesssim h^{k+2}\,|q|_{H^{\max(k+2,4)}(T)}, \quad \forall q \in H^{(k+2,4)}(T), \tag{4.99}$$

$$s_{\mathbf{curl},T}\left(\underline{I}^k_{\mathbf{curl},T}\boldsymbol{v}, \underline{I}^k_{\mathbf{curl},T}\boldsymbol{v}\right)^{1/2} \lesssim h^{k+1}\,|\boldsymbol{v}|_{\boldsymbol{H}^{\max(k+1,4)}(T)}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{(k+1,4)}(T), \tag{4.100}$$

$$s_{\nabla,T}\left(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}, \underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}\right)^{1/2} \lesssim h^{k+2}\,|\boldsymbol{w}|_{\boldsymbol{H}^{k+2}(T)}, \quad \forall \boldsymbol{w} \in \boldsymbol{H}^{k+2}(T), \tag{4.101}$$

$$s_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{W}, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{W}\right)^{1/2} \lesssim h^{k+1}\,|\boldsymbol{W}|_{\boldsymbol{H}^{(k+1,2)}(T)}, \quad \forall \boldsymbol{W} \in \boldsymbol{H}^{\max(k+1,2)}(T) \cap \boldsymbol{C}^0(\overline{T}). \tag{4.102}$$

Proof. Proof of (4.99), (4.100) and (4.101). We only prove (4.101), the other being established in a similar way. For all $\boldsymbol{z}_T \in \boldsymbol{\mathcal{P}}^{k+1}(T)$ we have $P^{k+1}_{\nabla,T}(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{z}_T) = \boldsymbol{z}_T$ by (4.72) and $\gamma^{k+2}_{\nabla,F}(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{z}_T) = \boldsymbol{z}_T$ by (4.66) so for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$,

$$\begin{aligned}
\mathrm{s}_{\nabla,T}\left(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{z}_T, \underline{\boldsymbol{v}}_T\right) &= \sum_{F\in\mathcal{F}_T} h_F \int_F (P^{k+1}_{\nabla,T}\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{z}_T - \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{z}_T)\cdot(P^{k+1}_{\nabla,T}\underline{\boldsymbol{v}}_T - \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{v}}_F) \\
&= \sum_{E\in\mathcal{E}_F} h_E^2 \int_E (P^{k+1}_{\nabla,T}\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{z}_T - \boldsymbol{z}_T)\cdot(P^{k+1}_{\nabla,T}\underline{\boldsymbol{v}}_T - \boldsymbol{v}_E) = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{s}_{\nabla,T}\left(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}, \underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{w}\right) &= \mathrm{s}_{\nabla,T}\left(\underline{\boldsymbol{I}}^k_{\nabla,T}(\boldsymbol{w} - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w}), \underline{\boldsymbol{I}}^k_{\nabla,T}(\boldsymbol{w} - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w})\right) \\
&\lesssim \|\underline{\boldsymbol{I}}^k_{\nabla,T}(\boldsymbol{w} - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{w})\|^2_{\nabla,T}.
\end{aligned}$$

We conclude by the equivalence of local norms (Lemma 4.25) and [59, Theorem 1.45].
Proof of (4.102). Let $\boldsymbol{W} \in \boldsymbol{H}^{k+1}(T) \cap \boldsymbol{C}^0(\overline{T})$, we have:

$$\begin{aligned}
\sum_{F\in\mathcal{F}_T} h_F \left\|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\boldsymbol{W})_{\otimes t,F} - \boldsymbol{\pi}^{k+1}_{\widetilde{\boldsymbol{\mathcal{P}}},F}(\boldsymbol{W})\right\|^2_F &\le \sum_{F\in\mathcal{F}_T} h_F \left\|(I - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T})\boldsymbol{W}\right\|^2_F \\
&\le \sum_{F\in\mathcal{F}_T} h_F \Big(\left\|\boldsymbol{W} - \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},T}\boldsymbol{W}\right\|^2_F \\
&\qquad + \left\|\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},T}\boldsymbol{W} - \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}\boldsymbol{W}\right\|^2_F\Big) \\
&\lesssim h^{2(k+1)}\,|\boldsymbol{W}|_{\boldsymbol{H}^{k+1}(T)} + \left\|\boldsymbol{W} - \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},T}\boldsymbol{W}\right\|^2_T.
\end{aligned}$$

Here the second inequality comes from $\overline{\boldsymbol{\mathcal{RT}}}^{k+1}(T)_{\otimes t,F} \subset \widetilde{\boldsymbol{\mathcal{P}}}^{k+1}(F)$ and we used the approximation properties on traces [59, Theorem 1.45 and Equation 1.75] on the first term and the discrete trace inequality [59, Lemma 1.32] on the second term to get the last inequality. We conclude with [59, Theorem 1.45 and Equation 1.74]. Likewise, we have:

$$\sum_{F\in\mathcal{F}_T} h_F \sum_{E\in\mathcal{E}_F} h_E \left\|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\boldsymbol{W})\,\boldsymbol{t}_E - \boldsymbol{\pi}^{k+2}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{W}\,\boldsymbol{t}_E)\right\|_E^2 \lesssim \sum_{F\in\mathcal{F}_T} h_F \sum_{E\in\mathcal{E}_F} h_E \left\|(I-\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T})\boldsymbol{W}\right\|_E^2.$$

In the case $k \geq 1$ we can proceed in the same way. However we must use the trace inequality twice to get a version of [59, Theorem 1.45] working on edges, which requires $k > 0$. For the case $k = 0$ we use a direct proof similar to [67, Theorem 6]. □

4.5.1 Poincaré inequality.

We begin by stating a lemma which will be useful to prove the Poincare inequality.

Lemma 4.31 (Estimates on local $\boldsymbol{H}^1$-seminorms of potentials)**.** *For all $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$ and all $\underline{\boldsymbol{w}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ it holds that*

$$\begin{aligned}\left\|\nabla\,\gamma^{k+2}_{\nabla,F}\,\underline{\boldsymbol{w}}_F\right\|_F^2 + \sum_{E\in\mathcal{E}_F} h_E^{-1}\left\|\gamma^{k+2}_{\nabla,F}\,\underline{\boldsymbol{w}}_F - \boldsymbol{w}_E\right\|_E^2 &\lesssim \left|\!\left|\!\left|\underline{\nabla}^{k+1}_F\,\underline{\boldsymbol{v}}_F\right|\!\right|\!\right|^2_{\boldsymbol{L}^2,F},\\ \left\|\nabla\,P^{k+1}_{\nabla,T}\,\underline{\boldsymbol{w}}_T\right\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1}\left\|P^{k+1}_{\nabla,T}\,\underline{\boldsymbol{w}}_T - \gamma^{k+2}_{\nabla,F}\,\underline{\boldsymbol{w}}_F\right\|_F^2 &\lesssim \left|\!\left|\!\left|\underline{\nabla}^{k+1}_T\,\underline{\boldsymbol{v}}_T\right|\!\right|\!\right|^2_{\boldsymbol{L}^2,T}.\end{aligned}\tag{4.103}$$

Proof. The proof is a simple adaptation of [67, Lemma 5.7]. □

Lemma 4.32 (Poincaré inequality)**.** *For all $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$ such that $\sum_{T\in\mathcal{T}_h}\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T = 0$ it holds that*

$$|\!|\!|\underline{\boldsymbol{w}}_h|\!|\!|_{\nabla,h} \lesssim \left|\!\left|\!\left|\underline{\nabla}^{k+1}_h\,\underline{\boldsymbol{w}}_h\right|\!\right|\!\right|_{\boldsymbol{L}^2,h}.\tag{4.104}$$

Proof. The proof is a simple adaptation of [67, Theorem 5.3]. □

Remark 4.33. When $k \geq 1$ the assumption $\sum_{T\in\mathcal{T}_h}\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T = 0$ translates to $\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{w}_T = 0$ by (4.73). However this does not hold when $k = 0$.

Lastly we show that the fully discrete divergence is onto $\underline{D}^k_h : \underline{\boldsymbol{X}}^k_{\nabla,h} \to \mathcal{P}^k(\mathcal{T}_h)$. The main difficulty is to show the boundedness of the inverse with the discrete norms. This requires a stronger assumption of quasi-uniformity on the mesh: $\forall T \in \mathcal{T}_h$, $h \lesssim h_T$.

Lemma 4.34 (Right-inverse for the divergence)**.** *If the mesh is quasi-uniform then for all $\underline{p}_h \in \mathcal{P}^k(\mathcal{T}_h)$ there is $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$ such that $\underline{D}^k_h\underline{\boldsymbol{w}}_h = \underline{p}_h$ and $|\!|\!|\underline{\boldsymbol{w}}_h|\!|\!|_{\nabla,h} + \left|\!\left|\!\left|\underline{\nabla}^{k+1}_h\underline{\boldsymbol{w}}_h\right|\!\right|\!\right|_{\boldsymbol{L}^2,h} \lesssim \|\underline{p}_h\|_{L^2,h}$.*

Proof. Existence. Let $\underline{p}_h = (p_T)_{T\in\mathcal{T}_h} \in \mathcal{P}^k(\mathcal{T}_h)$. Lemma 4.49 provides $\tilde{p} \in C^1(\overline{\Omega})$ such that $\forall T \in \mathcal{T}_h$, $\tilde{p}_{|T} \in \mathcal{P}^{k+\max_{h,T\in\mathcal{T}_h}(2|\mathcal{F}_T|)}(T)$, $\pi^k_{\mathcal{P},T}\tilde{p} = p_T$ and $\|\tilde{p}\|_{L^2(\Omega)} \approx \|\underline{p}_h\|_{L^2,h}$. Under the assumption on the regularity of the mesh sequence we have $\max_{h,T\in\mathcal{T}_h}(|\mathcal{F}_T|) \lesssim 1$ ([59, Lemma 1.12]) so that the maximum degree is bounded independently of h. Since $\tilde{p}$ is a piecewise polynomial, continuous, of continuous derivative and of trace zero on the boundary, $\tilde{p} \in H^2_0(\Omega)$. We apply Theorem 4.52 to find $\boldsymbol{u} \in \boldsymbol{H}^3(\Omega)$ such that $\operatorname{div}\boldsymbol{u} = \tilde{p}$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^3} \lesssim \|\tilde{p}\|_{H^2}$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^2} \lesssim \|\tilde{p}\|_{H^1}$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^1} \lesssim \|\tilde{p}\|_{L^2}$. Let $\underline{\boldsymbol{w}}_h = \underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}$, the commutation property (4.93c) gives $\underline{D}^k_h\underline{\boldsymbol{w}}_h = \underline{p}_h$.

Boundedness. Many bounds follow directly from the L^2-boundedness of the interpolator. Indeed, we make use of the continuous trace inequality [59, Lemma 1.31] to get

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{H}^k(F)} &\lesssim h^{-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^k(T)} + h^{\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(T)},\\
\|\boldsymbol{u}\|_{\boldsymbol{H}^k(E)} &\lesssim h^{-1}\|\boldsymbol{u}\|_{\boldsymbol{H}^k(T)} + \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(T)} + h\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+2}(T)}.
\end{aligned}$$

Using the estimate on $\|\boldsymbol{u}\|_{\boldsymbol{H}^k}$ and the discrete inverse inequality (Lemma 4.2) we get

$$\begin{aligned}
\sum_{F\in\mathcal{F}_h}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^k(F)} &\lesssim h^{1-2k}\|\underline{p}_h\|^2_{L^2,h}, \quad k=1,2,\\
\sum_{E\in\mathcal{E}_h}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(E)} &\lesssim h^{-2}\|\underline{p}_h\|^2_{L^2,h}.
\end{aligned} \tag{4.105}$$

This allows to bound all terms of $|||\underline{\boldsymbol{w}}_h|||_{\nabla,h}$.

It remains to estimate

$$|||\underline{\nabla}^{k+1}_h\underline{\boldsymbol{w}}_h|||^2_{\boldsymbol{L}^2,h} \approx \sum_{T\in\mathcal{T}_h}\left\|\nabla^{k+1}_T\underline{\boldsymbol{w}}_T\right\|^2_T + \sum_{F\in\mathcal{F}_h}h_F\left\|\nabla^{k+1}_F\underline{\boldsymbol{w}}_F\right\|^2_F + \sum_{E\in\mathcal{E}_h}h^2_E\left\|\nabla^{k+2}_E\underline{\boldsymbol{w}}_E\right\|^2_E.$$

The local commutation property (4.93c) and (4.105) give

$$\begin{aligned}
h^2\sum_{E\in\mathcal{E}_h}\left\|\nabla^{k+2}_E\underline{\boldsymbol{I}}^k_{\nabla,E}\boldsymbol{u}\right\|^2_E &= h^2\sum_{E\in\mathcal{E}_h}\left\|\boldsymbol{\pi}^{k+2}_{\boldsymbol{\mathcal{P}},E}(\nabla\boldsymbol{u}\,\boldsymbol{t}_E)\right\|^2_E \lesssim h^2\sum_{E\in\mathcal{E}_h}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(E)} \lesssim \|\underline{p}_h\|^2_{L^2,h},\\
h\sum_{F\in\mathcal{F}_h}\left\|\nabla^{k+1}_F\underline{\boldsymbol{I}}^k_{\nabla,F}\boldsymbol{u}\right\|^2_F &= h\sum_{F\in\mathcal{F}_h}\left\|\boldsymbol{\pi}^{k+1}_{\tilde{\boldsymbol{\mathcal{P}}},F}(\nabla\boldsymbol{u})\right\|^2_F \lesssim h\sum_{F\in\mathcal{F}_h}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(F)} \lesssim \|\underline{p}_h\|^2_{L^2,h},\\
\sum_{T\in\mathcal{T}_h}\left\|\nabla^{k+1}_T\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{u}\right\|^2_T &= \sum_{T\in\mathcal{T}_h}\left\|\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(\nabla\boldsymbol{u})\right\|^2_T \lesssim \sum_{T\in\mathcal{T}_h}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(T)} \lesssim \|\underline{p}_h\|^2_{L^2,h}.
\end{aligned}$$

□

Remark 4.35. We can easily adapt Lemma 4.34 to require $\sum_{T\in\mathcal{T}_h}\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T = 0$. Defining $\underline{\boldsymbol{w}'}_h = \underline{\boldsymbol{w}}_h - \underline{\boldsymbol{I}}^k_{\nabla,h}\left(\frac{1}{\int_\Omega 1}\sum_{T\in\mathcal{T}_h}\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right)$, it is clear from (4.72) that

$\sum_{T\in\mathcal{T}_h}\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}'}_T = 0$, from the local commutation properties (Lemma 4.26) that $\underline{D}^k_h\underline{\boldsymbol{w}'}_h = \underline{p}_h$ and from (4.87) that the estimate of Lemma 4.34 on the norm of $\underline{\boldsymbol{w}'}_h$ still holds.

4.5.2 Adjoint consistency.

We define the adjoint consistency error for the gradient for all $\boldsymbol{v}\in\boldsymbol{C}^1(\overline{\Omega})\cap\boldsymbol{H}_0(\mathrm{div},\Omega)$ and all $\underline{q}_h\in\underline{X}^k_{\mathbf{grad},h}$ by:

$$\tilde{\mathcal{E}}_{\mathbf{grad},h}(\boldsymbol{v},\underline{q}_h)=\sum_{T\in\mathcal{T}_h}\left(\left(\underline{\boldsymbol{I}}^k_{\mathbf{curl},T}\boldsymbol{v},\underline{\boldsymbol{G}}^k_T\underline{q}_T\right)_{\mathbf{curl},T}+\int_T\mathrm{div}\,\boldsymbol{v}\;P^{k+1}_{\mathbf{grad},T}\underline{q}_T\right).\quad(4.106)$$

Theorem 4.36 (Adjoint consistency for the gradient)**.** *For all $\boldsymbol{v}\in\boldsymbol{C}^1(\overline{\Omega})\cap\boldsymbol{H}_0(\mathrm{div},\Omega)$ such that $\boldsymbol{v}\in\boldsymbol{H}^{\max(k+1,4)}(\mathcal{T}_h)$ and all $\underline{q}_h\in\underline{X}^k_{\mathbf{grad},h}$, it holds:*

$$\left|\tilde{\mathcal{E}}_{\mathbf{grad},h}(\boldsymbol{v},\underline{q}_h)\right|\lesssim h^{k+1}\left|\boldsymbol{v}\right|_{\boldsymbol{H}^{(k+1,4)}(\mathcal{T}_h)}\left|\!\left|\!\left|\underline{\boldsymbol{G}}^k_h\underline{q}_h\right|\!\right|\!\right|_{\mathbf{curl},h}.\quad(4.107)$$

Proof. The proof is similar to [67, Section 6.4]. □

We define the adjoint consistency error for the curl for all $\boldsymbol{v}\in\boldsymbol{C}^0(\overline{\Omega})\cap\boldsymbol{H}_0(\mathbf{curl},\Omega)$ and all $\underline{\boldsymbol{w}}_h\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$ by:

$$\tilde{\mathcal{E}}_{\mathbf{curl},h}(\boldsymbol{v},\underline{\boldsymbol{w}}_h)=\sum_{T\in\mathcal{T}_h}\left(\left(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{v},\underline{\boldsymbol{C}}^k_T\underline{\boldsymbol{w}}_T\right)_{\nabla,T}-\int_T\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{P}^k_{\mathbf{curl},T}\underline{\boldsymbol{w}}_T\right).\quad(4.108)$$

Theorem 4.37 (Adjoint consistency for the curl)**.** *For all $\boldsymbol{v}\in\boldsymbol{C}^1(\overline{\Omega})\cap\boldsymbol{H}_0(\mathbf{curl},\Omega)$ such that $\boldsymbol{v}\in\boldsymbol{H}^{k+2}(\mathcal{T}_h)$ and all $\underline{\boldsymbol{w}}_h\in\underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$, it holds:*

$$\left|\tilde{\mathcal{E}}_{\mathbf{curl},h}(\boldsymbol{v},\underline{\boldsymbol{w}}_h)\right|\lesssim h^{k+1}\left(\left|\boldsymbol{v}\right|_{\boldsymbol{H}^{k+1}(\mathcal{T}_h)}+\left|\boldsymbol{v}\right|_{\boldsymbol{H}^{k+2}(\mathcal{T}_h)}\right)\left(\left|\!\left|\!\left|\underline{\boldsymbol{w}}_h\right|\!\right|\!\right|_{\mathbf{curl},h}+\left|\!\left|\!\left|\underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{w}}_h\right|\!\right|\!\right|_{\nabla,h}\right).\quad(4.109)$$

Proof. The proof uses the lifting from [67, Section 6.5]. Since the degrees of freedom not involved in this lifting only appear in the stabilization forms inside $\tilde{\mathcal{E}}_{\mathbf{curl},h}$, we can use (4.100) and conclude using similar arguments. □

The adjoint consistency error for ∇ is defined for all $\boldsymbol{V}\in\boldsymbol{C}^0(\overline{\Omega})\cap\boldsymbol{H}^1_0(\Omega)$ and all $\underline{\boldsymbol{w}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h}$ by:

$$\tilde{\mathcal{E}}_{\nabla,h}(\boldsymbol{V},\underline{\boldsymbol{w}}_h)=\sum_{T\in\mathcal{T}_h}\left(\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{V},\underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T\right)_{\boldsymbol{L}^2,T}+\int_T\nabla\cdot\boldsymbol{V}\cdot P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right).\quad(4.110)$$

Theorem 4.38 (Adjoint consistency for ∇)**.** *For all $\boldsymbol{V} \in \boldsymbol{C}^0(\overline{\Omega}) \cap \boldsymbol{H}^1_0(\Omega)$ such that $\boldsymbol{V} \in \boldsymbol{H}^{k+2}(\mathcal{T}_h)$ and all $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$, it holds:*

$$\left|\tilde{\mathcal{E}}_{\nabla,h}(\boldsymbol{V},\underline{\boldsymbol{w}}_h)\right| \lesssim h^{k+1}\left(|\boldsymbol{V}|_{\boldsymbol{H}^{k+1}} + |\boldsymbol{V}|_{\boldsymbol{H}^{k+2}}\right)\left(|||\underline{\boldsymbol{w}}_h|||_{\nabla,h} + \left|\left|\left|\underline{\nabla}^{k+1}_h \underline{\boldsymbol{w}}_h\right|\right|\right|_{\boldsymbol{L}^2,h}\right). \tag{4.111}$$

Proof. Remarks 4.19 and 4.5 show that $\forall \boldsymbol{V}_h \in (\boldsymbol{\mathcal{RT}}^{k+1}(\mathcal{T}_h)^\intercal)^3$, $\forall T \in \mathcal{T}_h$,

$$\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T \cdot \nabla\cdot \boldsymbol{V}_T + \int_T \nabla^{k+1}_T \underline{\boldsymbol{w}}_T : \boldsymbol{V}_T - \sum_{F\in\mathcal{F}_T} \omega_{TF}\int_F \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F \boldsymbol{V}_T \boldsymbol{n}_F = 0.$$

Let $\widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F$ be the continuous polynomial given by Lemma 4.49 such that $\boldsymbol{\pi}^{k+2}_{\boldsymbol{\mathcal{P}},F}(\widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F) = \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F$, $(\widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F)_{|E} = \boldsymbol{0}$, $(\nabla\, \widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F)_{|E} = \boldsymbol{0}$ and $\left\|\widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F\right\|_F \approx \left\|\gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F\right\|_F$. It holds that

$$\sum_{F\in\mathcal{F}_T} \omega_{TF}\int_F \gamma^{k+2}_{\nabla,F}\underline{\boldsymbol{w}}_F \boldsymbol{V}_T\boldsymbol{n}_F = \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F \boldsymbol{V}_T \boldsymbol{n}_F.$$

Moreover, since $\boldsymbol{V}\cdot\boldsymbol{n}_\Omega = 0$ on $\partial\Omega$ and since the $\widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \boldsymbol{w}_F$ are single valued we have

$$\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F \boldsymbol{V}\boldsymbol{n}_F = 0. \tag{4.112}$$

Hence we can write:

$$\begin{aligned}
\tilde{\mathcal{E}}_{\nabla,h}(\boldsymbol{V},\underline{\boldsymbol{w}}_h) = \sum_{T\in\mathcal{T}_h}\Bigg(&\int_T(\boldsymbol{V}-\boldsymbol{V}_T) : \nabla^{k+1}_T \underline{\boldsymbol{w}}_T + \int_T \nabla\cdot(\boldsymbol{V}-\boldsymbol{V}_T)\cdot P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T \\
&+ \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_E \widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F(\boldsymbol{V}_T - \boldsymbol{V})\boldsymbol{n}_F + \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{V}, \nabla^{k+1}_T \underline{\boldsymbol{w}}_T\right)\Bigg) \\
\lesssim \sum_{T\in\mathcal{T}_h} &\left(\|\boldsymbol{V}-\boldsymbol{V}_T\| + \|\nabla\cdot(\boldsymbol{V}-\boldsymbol{V}_T)\|\right)\left(\left\|\nabla^{k+1}_T \underline{\boldsymbol{w}}_T\right\| + \left\|P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T\right\|\right) \\
&+ \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{V}, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{V}\right)^{1/2} \mathrm{s}_{\boldsymbol{L}^2,T}\left(\nabla^{k+1}_T \underline{\boldsymbol{w}}_T, \nabla^{k+1}_T \underline{\boldsymbol{w}}_T\right)^{1/2} \\
&+ \sum_{F\in\mathcal{F}_T}\omega_{TF}\int_F \widetilde{\gamma^{k+2}_{\nabla,F}}^c\, \underline{\boldsymbol{w}}_F(\boldsymbol{V}_T-\boldsymbol{V})\boldsymbol{n}_F.
\end{aligned}$$

Applying (4.102) and the equivalence of local norms (Lemma 4.25) gives:

$$\begin{aligned}
\mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{V}, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\boldsymbol{V}\right)^{1/2} &\mathrm{s}_{\boldsymbol{L}^2,T}\left(\nabla^{k+1}_T \underline{\boldsymbol{w}}_T, \nabla^{k+1}_T \underline{\boldsymbol{w}}_T\right)^{1/2} \\
&\lesssim h^{k+1}\,|\boldsymbol{V}|_{\boldsymbol{H}^{(k+1,2)}(T)} \left|\left|\left|\underline{\nabla}^{k+1}_T \underline{\boldsymbol{w}}_T\right|\right|\right|_{\boldsymbol{L}^2,T}.
\end{aligned}$$

Using the approximation properties of the spaces $\boldsymbol{\mathcal{RT}}^{k+1}(T)$ given by a slight adaptation of [67, Lemma 6.8] we can find $\boldsymbol{V}_T \in (\boldsymbol{\mathcal{RT}}^{k+1}(T)^\intercal)^3$ such that

$$\|\boldsymbol{V} - \boldsymbol{V}_T\| + \|\nabla\cdot(\boldsymbol{V} - \boldsymbol{V}_T)\| \lesssim h^{k+1} \left(|\boldsymbol{V}|_{\boldsymbol{H}^{k+1}(T)} + |\boldsymbol{V}|_{\boldsymbol{H}^{k+2}(T)}\right)$$

By (4.87) we see that

$$\left\|\nabla_T^{k+1} \underline{\boldsymbol{w}}_T\right\| + \left\|P_{\nabla,T}^{k+1}\underline{\boldsymbol{w}}_T\right\| \lesssim \left|\!\left|\!\left|\underline{\nabla}_T^{k+1}\underline{\boldsymbol{w}}_T\right|\!\right|\!\right|_{\boldsymbol{L}^2,T} + |\!|\!|\underline{\boldsymbol{w}}_T|\!|\!|_{\nabla,T}.$$

Lastly we use Theorem 4.50 to find $\boldsymbol{R}_{\underline{\boldsymbol{w}}_T} \in \boldsymbol{H}^1(T)$ such that

$$\begin{aligned}\sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \widetilde{\gamma_{\nabla,F}^{k+2}}^c \underline{\boldsymbol{w}}_F(\boldsymbol{V}_T - \boldsymbol{V})\boldsymbol{n}_F &= \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \boldsymbol{R}_{\underline{\boldsymbol{w}}_T}(\boldsymbol{V}_T - \boldsymbol{V})\boldsymbol{n}_F \\ &= \int_T \nabla \boldsymbol{R}_{\underline{\boldsymbol{w}}_T} :(\boldsymbol{V}_T - \boldsymbol{V}) + \int_T \boldsymbol{R}_{\underline{\boldsymbol{w}}_T} \cdot \nabla\cdot(\boldsymbol{V}_T - \boldsymbol{V}).\end{aligned}$$

Hence

$$\begin{aligned}\left|\sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \widetilde{\gamma_{\nabla,F}^{k+2}}^c \underline{\boldsymbol{w}}_F(\boldsymbol{V}_T - \boldsymbol{V})\boldsymbol{n}_F\right| &\lesssim (\|\boldsymbol{V} - \boldsymbol{V}_T\| + \|\nabla\cdot(\boldsymbol{V}_T - \boldsymbol{V})\|) \\ &\quad\times \left(\left\|\nabla \boldsymbol{R}_{\underline{\boldsymbol{w}}_T}\right\| + \left\|\boldsymbol{R}_{\underline{\boldsymbol{w}}_T}\right\|\right),\end{aligned}$$

and we conclude with Theorem 4.50 which gives the boundedness of $\boldsymbol{R}_{\underline{\boldsymbol{w}}_T}$. □

We can sharpen the estimate (4.110) when $\boldsymbol{V}$ is the gradient of some field. Indeed, if we were to take $\boldsymbol{V} = \nabla v$ in the adjoint consistency (Theorem 4.38), we would see that a norm over $\boldsymbol{H}^{k+3}$ appears in the estimate, which is not optimal.

We define the adjoint consistency error for all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$ such that $\nabla \boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0$ and all $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$ by:

$$\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{v}, \underline{\boldsymbol{w}}_h) = \sum_{T\in\mathcal{T}_h} \left(\int_T \boldsymbol{\Delta}\, \boldsymbol{v} \cdot P_{\nabla,T}^{k+1}\underline{\boldsymbol{w}}_T + \left(\underline{\nabla}_T^{k+1}\underline{\boldsymbol{I}}_{\nabla,T}^k\boldsymbol{v}, \underline{\nabla}_T^{k+1}\underline{\boldsymbol{w}}_T\right)_{\boldsymbol{L}^2,T}\right). \quad (4.113)$$

Theorem 4.39 (Adjoint consistency for the Laplacian)**.** *For all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{C}^1(\overline{\Omega})$ such that $\nabla \boldsymbol{v} \cdot \boldsymbol{n}_\Omega = 0$ and $\boldsymbol{v} \in \boldsymbol{H}^{k+2}(\mathcal{T}_h)$ and for all $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}_{\nabla,h}^k$, it holds:*

$$\left|\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{v}, \underline{\boldsymbol{w}}_h)\right| \lesssim h^{k+1} |\boldsymbol{v}|_{\boldsymbol{H}^{(k+2,3)}} \left|\!\left|\!\left|\underline{\nabla}_h^{k+1}\underline{\boldsymbol{w}}_h\right|\!\right|\!\right|_{\boldsymbol{L}^2,h}. \quad (4.114)$$

Proof. For any $T \in \mathcal{T}_h$, (4.93c) gives:

$$\left(\underline{\nabla}_T^{k+1}\underline{\boldsymbol{I}}_{\nabla,T}^k\boldsymbol{v}, \underline{\nabla}_T^{k+1}\underline{\boldsymbol{w}}_T\right)_{\boldsymbol{L}^2,T} = \int_T \boldsymbol{\pi}_{\boldsymbol{\mathcal{RT}},T}^{k+1} \nabla \boldsymbol{v} : \nabla_T^{k+1} \underline{\boldsymbol{w}}_T + \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}_{\boldsymbol{L}^2,T}^k \nabla \boldsymbol{v}, \underline{\nabla}_T^{k+1}\underline{\boldsymbol{w}}\right).$$

With an integration by parts and since $\int_T \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T} \nabla \boldsymbol{v} : \nabla^{k+1}_T \underline{\boldsymbol{w}}_T = \int_T \nabla \boldsymbol{v} : \nabla^{k+1}_T \underline{\boldsymbol{w}}_T$ we have:

$$\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{v}, \underline{\boldsymbol{w}}_h) = \sum_{T\in\mathcal{T}_h} \left(\int_T \nabla \boldsymbol{v} : (\nabla^{k+1}_T \underline{\boldsymbol{w}}_T - \nabla P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T) + \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T} \nabla \boldsymbol{v}, \underline{\nabla}^{k+1}_T \underline{\boldsymbol{w}} \right) \right.$$
$$\left. + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T \nabla \boldsymbol{v}\, \boldsymbol{n}_F \right).$$

Since we assume $\nabla \boldsymbol{v} \cdot \boldsymbol{n}_\Omega = 0$ we have

$$\sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \nabla \boldsymbol{v}\, \boldsymbol{n}_F = 0. \tag{4.115}$$

By Remark 4.19 it holds $\forall \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^{k+1}(T)$,

$$\int_T \boldsymbol{\Delta}\, \boldsymbol{v}_T \cdot P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T + \int_T \nabla^{k+1}_T \underline{\boldsymbol{w}}_T : \nabla \boldsymbol{v}_T - \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \nabla \boldsymbol{v}_T\, \boldsymbol{n}_F = 0,$$

so

$$\int_T \nabla \boldsymbol{v}_T : (\nabla^{k+1}_T \underline{\boldsymbol{w}}_T - \nabla P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T) + \sum_{F\in\mathcal{F}_T} \omega_{TF} \int_F (P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T - \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F) \nabla \boldsymbol{v}_T\, \boldsymbol{n}_F = 0.$$

This allows us to write for any $\underline{\boldsymbol{v}}_h = (\boldsymbol{v}_T)_{T\in\mathcal{T}_h} \in \boldsymbol{\mathcal{P}}^{k+1}(\mathcal{T}_h)$,

$$\begin{aligned}
\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{v}, \underline{\boldsymbol{w}}_h) &= \sum_{T\in\mathcal{T}_h} \left(\int_T \nabla(\boldsymbol{v} - \boldsymbol{v}_T) : (\nabla^{k+1}_T \underline{\boldsymbol{w}}_T - \nabla P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T) \right. \\
&\quad + \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T} \nabla \boldsymbol{v}, \underline{\nabla}^{k+1}_T \underline{\boldsymbol{w}} \right) \\
&\quad \left. + \sum_{F\in\mathcal{F}_T} \omega_{FE} \int_F (P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T - \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F) \nabla(\boldsymbol{v} - \boldsymbol{v}_T) \boldsymbol{n}_F \right), \\
\left| \tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{v}, \underline{\boldsymbol{w}}_h) \right| &\lesssim \sum_{T\in\mathcal{T}_h} \left(\|\nabla(\boldsymbol{v} - \boldsymbol{v}_T)\|_T \left\| \nabla^{k+1}_T \underline{\boldsymbol{w}}_T - \nabla P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T \right\|_T \right. \\
&\quad + \sum_{F\in FT} \left\| P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T - \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \right\|_F \|\nabla(\boldsymbol{v} - \boldsymbol{v}_T)\|_F \\
&\quad \left. + \left| \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T} \nabla \boldsymbol{v}, \underline{\nabla}^{k+1}_T \underline{\boldsymbol{w}} \right) \right| \right).
\end{aligned}$$

Applying the estimates on the $\boldsymbol{H}^1$-seminorms of potentials (Lemma 4.31) we get

$$\left\| P^{k+1}_{\nabla,T} \underline{\boldsymbol{w}}_T - \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{w}}_F \right\|_F \|\nabla(\boldsymbol{v} - \boldsymbol{v}_T)\|_F \lesssim \left|\left|\left| \underline{\nabla}^{k+1}_T \underline{\boldsymbol{w}}_T \right|\right|\right|_{\boldsymbol{L}^2,T} h^{\frac{1}{2}} \|\nabla(\boldsymbol{v} - \boldsymbol{v}_T)\|_F .$$

Furthermore (4.102) and the equivalence of local norms (Lemma 4.25) give

$$\left|\mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}\,\nabla\,\boldsymbol{v}, \underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}\right)\right| \lesssim h^{k+1}\left|\nabla\,\boldsymbol{v}\right|_{\boldsymbol{H}^{(k+1,2)}} \left|\left|\left|\underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T\right|\right|\right|_{\boldsymbol{L}^2,T}.$$

Hence, applying again Lemma 4.31 we write:

$$\begin{aligned}\left|\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{v},\underline{\boldsymbol{w}}_h)\right| \lesssim \sum_{T\in\mathcal{T}_h}&\left[\left|\left|\left|\underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T\right|\right|\right|_{\boldsymbol{L}^2,T}\left(\left\|\nabla(\boldsymbol{v}-\boldsymbol{v}_T)\right\|_T + \sum_{F\in\mathcal{F}_T} h^{\frac{1}{2}}\left\|\nabla(\boldsymbol{v}-\boldsymbol{v}_T)\right\|_F\right)\right]\\ &+ h^{k+1}\left|\boldsymbol{v}\right|_{\boldsymbol{H}^{(k+2,3)}}\left|\left|\left|\underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_h\right|\right|\right|_{\boldsymbol{L}^2,T}.\end{aligned}$$

We conclude by taking $\boldsymbol{v}_T = \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{v}$, then the approximation properties [59, Theorem 1.45] give:

$$\left\|\boldsymbol{v}-\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{v}\right\|_{\boldsymbol{H}^1(T)} \lesssim h^{k+1}\left|\boldsymbol{v}\right|_{\boldsymbol{H}^{k+2}},$$

$$h^{\frac{1}{2}}\left\|\boldsymbol{v}-\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},T}\boldsymbol{v}\right\|_{\boldsymbol{H}^1(F)} \lesssim h^{k+1}\left|\boldsymbol{v}\right|_{\boldsymbol{H}^{k+2}}.$$

□

4.6 Stokes equations.

Finally, we illustrate this complex with the design of a scheme for the Stokes equations. For the sake of simplicity we use Neumann boundary conditions over the whole boundary, which amounts to a free outlet condition. More general conditions are not difficult to enforce. The solution is determined only up to a constant vector field. This leads to the introduction of a new space:

$$\underline{\boldsymbol{X}}^k_{\nabla,h,\star} := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h} \,:\, \sum_{T\in\mathcal{T}_h}\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{v}}_T = 0\}. \tag{4.116}$$

This is the discrete counterpart of $\boldsymbol{H}^1(\Omega)\cap L^2_0(\Omega)$.

Let μ be a constant viscosity, we define the symmetric bilinear form $\mathrm{a}_h : \underline{\boldsymbol{X}}^k_{\nabla,h}\times\underline{\boldsymbol{X}}^k_{\nabla,h}\to\mathbb{R}$ such that, for all $\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$,

$$\mathrm{a}_h(\underline{\boldsymbol{v}}_h,\underline{\boldsymbol{w}}_h) := \mu\left(\underline{\nabla}^{k+1}_h\underline{\boldsymbol{v}}_h, \underline{\nabla}^{k+1}_h\underline{\boldsymbol{w}}_h\right)_{\boldsymbol{L}^2,h}. \tag{4.117}$$

We also define the bilinear form $\mathrm{b}_h : \underline{\boldsymbol{X}}^k_{\nabla,h}\times\mathcal{P}^k(\mathcal{T}_h)\to\mathbb{R}$ such that, for all $\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{X}}^k_{\nabla,h}, \underline{q}_h\in\mathcal{P}^k(\mathcal{T}_h)$,

$$\mathrm{b}_h(\underline{\boldsymbol{v}}_h,\underline{q}_h) := \sum_{T\in\mathcal{T}_h}\int_T D^k_T\underline{\boldsymbol{v}}_T q_T. \tag{4.118}$$

Then we define the bilinear form $\mathcal{A}_h : (\underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)) \times (\underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)) \to \mathbb{R}$ such that, for all $(\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)$,

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) = \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h) - \mathrm{b}_h(\underline{\boldsymbol{w}}_h, \underline{p}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, \underline{q}_h). \tag{4.119}$$

We define a suitable Sobolev-like norm on our discrete spaces such that $\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$,

$$\|\underline{\boldsymbol{v}}_h\|_{\mu,\nabla,1,h} := \left(\mu\|\underline{\boldsymbol{v}}_h\|^2_{\nabla,h} + \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h)\right)^{1/2}. \tag{4.120}$$

And for $f \in \boldsymbol{L}^2(\Omega)$ we set $\mathcal{L}_h : \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \to \mathbb{R}$ such that $\forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h}$,

$$\mathcal{L}_h(\underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \int_T P^{k+1}_{\nabla,T} \underline{\boldsymbol{v}}_T \cdot f. \tag{4.121}$$

We define the discrete problem:
Find $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)$ such that for all $(\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)$,

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) = \mathcal{L}_h(\underline{\boldsymbol{v}}_h). \tag{4.122}$$

We show well-posedness in Lemma 4.41.

We consider the following Stokes problem:
Find $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{L}^2_0(\Omega)$, $p \in H^1(\Omega)$ such that

$$\begin{aligned} -\mu\,\boldsymbol{\Delta}\,\boldsymbol{u} + \mathbf{grad}\,p &= f, \text{ on } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0, \text{ on } \Omega, \\ -\mu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_\Omega} + p\boldsymbol{n}_\Omega &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{4.123}$$

Let $(\boldsymbol{u}, p)$ solves (4.123) and let $(\underline{\boldsymbol{v}}_h, \underline{p}_h)$ solves (4.122). We assume that the continuous solutions $\boldsymbol{u}$ and p have the additional smoothness $\boldsymbol{u} \in \boldsymbol{H}^{k+2}(\mathcal{T}_h)$ and $p \in H^{k+2}(\mathcal{T}_h)$. We also assume the quasi-uniformity of the mesh in order to use Remark 4.35. We deduce the following error estimate.

Theorem 4.40 (Error estimate for Stokes)**.** *Under the smoothness assumption on $\boldsymbol{u}$ and p it holds that*

$$\begin{aligned} &\|\underline{\boldsymbol{v}}_h - \underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}\|_{\mu,\nabla,1,h} + \mu^{-\frac{1}{2}}\|\underline{p}_h - \underline{\pi}^k_{P,\mathcal{T}_h} p\|_{L^2,h} \\ &\qquad \lesssim h^{k+1}\left(\mu^{\frac{1}{2}}\,|\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(\mathcal{T}_h)} + \mu^{-\frac{1}{2}}\,|p|_{H^{k+1}(\mathcal{T}_h)} + \mu^{-\frac{1}{2}}\,|p|_{H^{k+2}(\mathcal{T}_h)}\right). \end{aligned} \tag{4.124}$$

Proof. The proof is a direct application of the third Strang lemma (see [52]) to the estimates given by Lemma 4.41 and 4.42. □

Lemma 4.41 (Well-posedness). *For any* $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)$ *there is* $(\underline{\boldsymbol{w}}_h, \underline{q}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)$ *such that* $\|\underline{\boldsymbol{w}}_h\|_{\mu,\nabla,1,h} + \mu^{-\frac{1}{2}}\|\underline{q}_h\|_{L^2,h} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\mu,\nabla,1,h} + \mu^{-\frac{1}{2}}\|\underline{p}_h\|_{L^2,h}$ *and*

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{w}}_h, \underline{q}_h)) \gtrsim \|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h} + \mu^{-1}\|\underline{p}_h\|^2_{L^2,h},$$

where hidden constants are independent of μ *and* h.

Proof. Let $(\underline{\boldsymbol{v}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h)$, we have

$$\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\underline{\boldsymbol{v}}_h, \underline{p}_h)) = \mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) \gtrsim \|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h}, \tag{4.125}$$

where the last inequality comes from Poincaré inequality (Lemma 4.32). Moreover by Remark 4.35 there is $\underline{\boldsymbol{w}'}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star}$ and $c > 0$ independent of μ and h such that $\underline{D}^k_h \underline{\boldsymbol{w}'}_h = -\underline{p}_h$ and $\|\underline{\boldsymbol{w}'}_h\|_{\mu,\nabla,1,h} \leq c\mu^{\frac{1}{2}}\|\underline{p}_h\|_{L^2,h}$. Hence

$$\begin{aligned}\mathcal{A}_h((\underline{\boldsymbol{v}}_h, \underline{p}_h), (\mu^{-1}\underline{\boldsymbol{w}'}_h, 0)) &= \mu^{-1}\mathrm{a}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}'}_h) + \mu^{-1}\|\underline{p}_h\|^2_{L^2,h} \\ &\geq -\frac{c^2}{2}\|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h} - \frac{1}{2c^2\mu^2}\|\underline{\boldsymbol{w}'}_h\|^2_{\mu,\nabla,1,h} + \mu^{-1}\|\underline{p}_h\|^2_{L^2,h} \\ &\geq -\frac{c^2}{2}\|\underline{\boldsymbol{v}}_h\|^2_{\mu,\nabla,1,h} + \frac{1}{2}\mu^{-1}\|\underline{p}_h\|^2_{L^2,h}.\end{aligned} \tag{4.126}$$

We conclude summing (4.125) and (4.126). □

We define the consistency error $\mathcal{E}_h : \underline{\boldsymbol{X}}^k_{\nabla,h,\star} \times \mathcal{P}^k(\mathcal{T}_h) \to \mathbb{R}$ by

$$\mathcal{E}_h((\underline{\boldsymbol{w}}_h, \underline{q}_h)) = \mathcal{L}_h(\underline{\boldsymbol{w}}_h) - \mathcal{A}_h((\underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}, \underline{\pi}^k_{P,\mathcal{T}_h}p), (\underline{\boldsymbol{w}}_h, \underline{q}_h)). \tag{4.127}$$

Lemma 4.42. *For all* $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star}$, $\underline{q}_h \in \mathcal{P}^k(\mathcal{T}_h)$, *it holds*

$$\mathcal{E}_h((\underline{\boldsymbol{w}}_h, \underline{q}_h)) \lesssim h^{k+1}\left(\mu^{\frac{1}{2}}\,|\boldsymbol{u}|_{\boldsymbol{H}^{(k+2,3)}(\mathcal{T}_h)} + \mu^{-\frac{1}{2}}\,|p|_{H^{k+1}(\mathcal{T}_h)} + \mu^{-\frac{1}{2}}\,|p|_{H^{k+2}(\mathcal{T}_h)}\right)\|\underline{\boldsymbol{w}}_h\|_{\mu,\nabla,1,h}.$$

Proof. Let $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{X}}^k_{\nabla,h,\star}$, $\underline{q}_h \in \mathcal{P}^k(\mathcal{T}_h)$, then

$$\begin{aligned}
\mathcal{E}_h((\underline{\boldsymbol{w}}_h, \underline{q}_h)) &= \sum_{T\in\mathcal{T}_h} \int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T \cdot f - \mu\left(\underline{\nabla}^{k+1}_T \underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{u}, \underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T\right)_{\nabla,T} \\
&\quad + \int_T D^k_T\underline{\boldsymbol{w}}_T\,\pi^k_{\mathcal{P},T}p - \int_T D^k_T(\underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{u})\,q_T \\
&= \sum_{T\in\mathcal{T}_h} \int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T \cdot \mathbf{grad}\,p + \int_T D^k_T\underline{\boldsymbol{w}}_T\,p \\
&\quad - \mu\left(\int_T P^{k+1}_{\nabla,T}\underline{\boldsymbol{w}}_T \cdot \boldsymbol{\Delta}\,\boldsymbol{u} + \left(\underline{\nabla}^{k+1}_T \underline{\boldsymbol{I}}^k_{\nabla,T}\boldsymbol{u}, \underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T\right)_{\nabla,T}\right) \\
&= \tilde{\mathcal{E}}_{\nabla,h}(p\,\mathbf{I}_{3,3}, \underline{\boldsymbol{w}}_h) - \mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}(p\,\mathbf{I}_{3,3})\right) - \mu\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{u}, \underline{\boldsymbol{w}}_h) \\
&\leq \left|\tilde{\mathcal{E}}_{\nabla,h}(p\,\mathbf{I}_{3,3}, \underline{\boldsymbol{w}}_h) - \mu\tilde{\mathcal{E}}_{\boldsymbol{\Delta},h}(\boldsymbol{u}, \underline{\boldsymbol{w}}_h)\right| \\
&\quad + \left|\mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T, \underline{\nabla}^{k+1}_T\underline{\boldsymbol{w}}_T\right)\right|^{1/2} \left|\mathrm{s}_{\boldsymbol{L}^2,T}\left(\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}(p\,\mathbf{I}_{3,3}), \underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,T}(p\,\mathbf{I}_{3,3})\right)\right|^{1/2}.
\end{aligned}$$

The second equality comes from $f = -\mu\,\boldsymbol{\Delta}\,\boldsymbol{u} + \mathbf{grad}\,p$, (4.93c) and $\operatorname{div}\boldsymbol{u} = 0$, and the third equality comes from (4.110), (4.113) as well as:

$$\begin{aligned}
\int_T D^k_T\underline{\boldsymbol{w}}_T\,\pi^k_{\mathcal{P},T}p = \int_T \mathrm{Tr}(\nabla^{k+1}_T\,\underline{\boldsymbol{w}}_T)\,p = \int_T \nabla^{k+1}_T\,\underline{\boldsymbol{w}}_T : (p\,\mathbf{I}_{3,3}) \\
= \int_T \nabla^{k+1}_T\,\underline{\boldsymbol{w}}_T : \boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{RT}},T}(p\,\mathbf{I}_{3,3}).
\end{aligned}$$

We do not have the boundary conditions required to use the adjoint consistency theorems (Theorem 4.38 and 4.39). However, it is fairly straightforward to adapt them for the boundary condition prescribed in (4.123) since, although (4.112) and (4.115) are no longer zero, their weighted sum is. We conclude with the estimates for the adjoint consistency and for the stabilization form (4.102). □

4.7 Numerical validation.

The Stokes complex was implemented within the HArDCore C++ framework (see `https://github.com/jdroniou/HArDCore`), using the linear algebra facilities from the Eigen3 library (see `https://eigen.tuxfamily.org`) and the linear solver from the Portable, Extensible Toolkit for Scientific Computation PETSc (see `https://petsc.org`). An implementation of the spaces and operators defined in this paper as well as a Stokes solver can be found at `https://github.com/mlhanot/HArDCore3D-Stokes`. The numerical validation is done with a constant viscosity $\mu = 1$. We measure the rate of convergence toward the exact solution:

$$\boldsymbol{u} = \begin{pmatrix} \pi x^2(1-x)^2 \sin(\pi y)\sin(\pi z)\cos(\pi y) \\ x(1-x)\sin(\pi y)^2\sin(\pi z)\cos(\pi z) - (2x^3-3x^2+x)\sin(\pi y)^2\sin(\pi z) \\ -x(1-x)\sin(\pi y)\sin(\pi z)^2\cos(\pi y) \end{pmatrix},$$
$$p = \sin(\frac{\pi x}{2}) - 8yz,$$

on the domain $\Omega = [0,1]^3$ for various polynomial degrees $k \in \{0,1,2,3\}$. The error is computed as

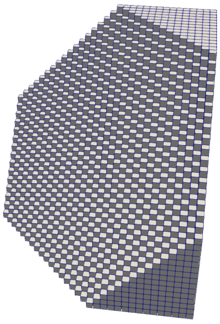
$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{u}\|_{\mu,\nabla,1,h} + \|\underline{p}_h - \underline{\pi}^k_{P,\mathcal{T}_h} p\|_{L^2,h}\,.$$

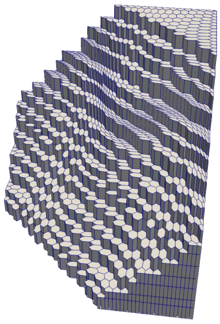
We expect the error to decrease at a rate $\mathcal{O}(h^{k+1})$ thanks to the error estimates for Stokes (Theorem 4.40). We also validated the 2-dimensional variation detailed in appendix 4.C. These tests are done on various mesh sequences. One mesh of each sequence is shown in Figure 4.3 for the 2-dimensional cases, and a cross section of the 3-dimensional meshes is shown in Figure 4.2. The results are given in Figure 4.4 in 2 dimensions and in Figure 4.5 in 3 dimensions. We always obtain results consistent with the theory. In 2 dimensions, we can see that the method is robust and the convergence is barely impacted by the various features of the meshes. In 3 dimensions, arbitrary polyhedra can be much wilder than arbitrary polygons. We can see that, while the expected convergence rate is asymptotically obtained, some combinations of degree and mesh exhibit better properties. On the coarsest meshes with the lowest polynomial degree we notice a drop in the convergence rate. These meshes are too coarse for the solution sought and the problem disappears when refining or increasing the polynomial degree. The case of degree 0 on tetrahedra might require further investigation as the observed rate reaches roughly 0.5. We also noticed some difficulty in using the preconditioner used for all other cases.

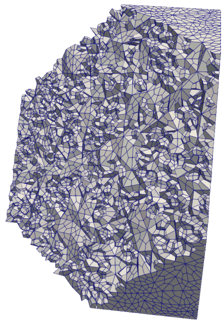
4.A Results on polynomial spaces.

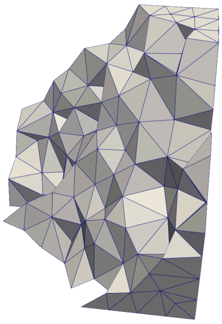
We begin by showing a few results to complete the introduction of spaces (4.16) and (4.18). Let $T \in \mathcal{T}_h$ be any cell. We assume without loss of generality that $\boldsymbol{x}_T = 0$, where $\boldsymbol{x}_T \in T$ is the point given by (4.7). We identify the coordinate with three variables x, y and z, and we define $\mathcal{P}^k[X,Y](T) = \{P \in \mathcal{P}^k(T),\ \deg_Z(P) = 0\}$. By (4.7) we can write:

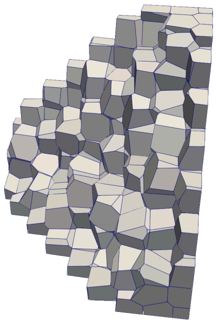
$$(\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3 = \begin{pmatrix} xP_1 & yP_1 & zP_1 \\ xP_2 & yP_2 & zP_2 \\ xP_3 & yP_3 & zP_3 \end{pmatrix},\ P_{i\in\{1,2,3\}} \in \mathcal{P}^{k-1}(T). \tag{4.128}$$


(a) "Cube".


(b) "Prismatic".


(c) "Hexahedra".


(d) "Tetgen cube".


(e) "Voro".

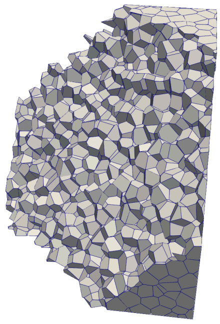
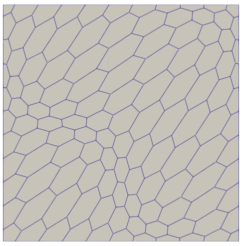
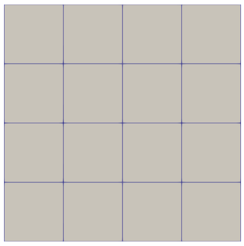
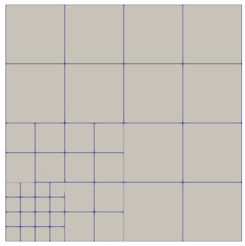
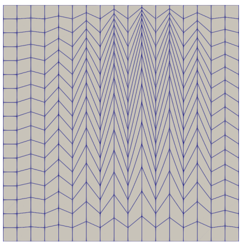

(f) "Voro tets".

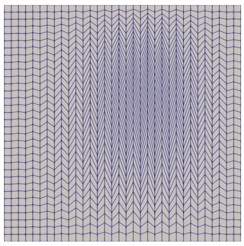
Figure 4.2: Families of meshes used in 3 dimensions, sliced for visualization.


(a) "Hexa".


(b) "Square".


(c) "Square 2".


(d) "Tilted".


(e) "Tilted 2".

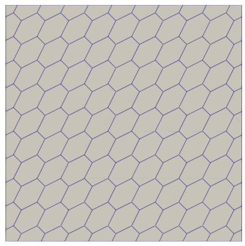
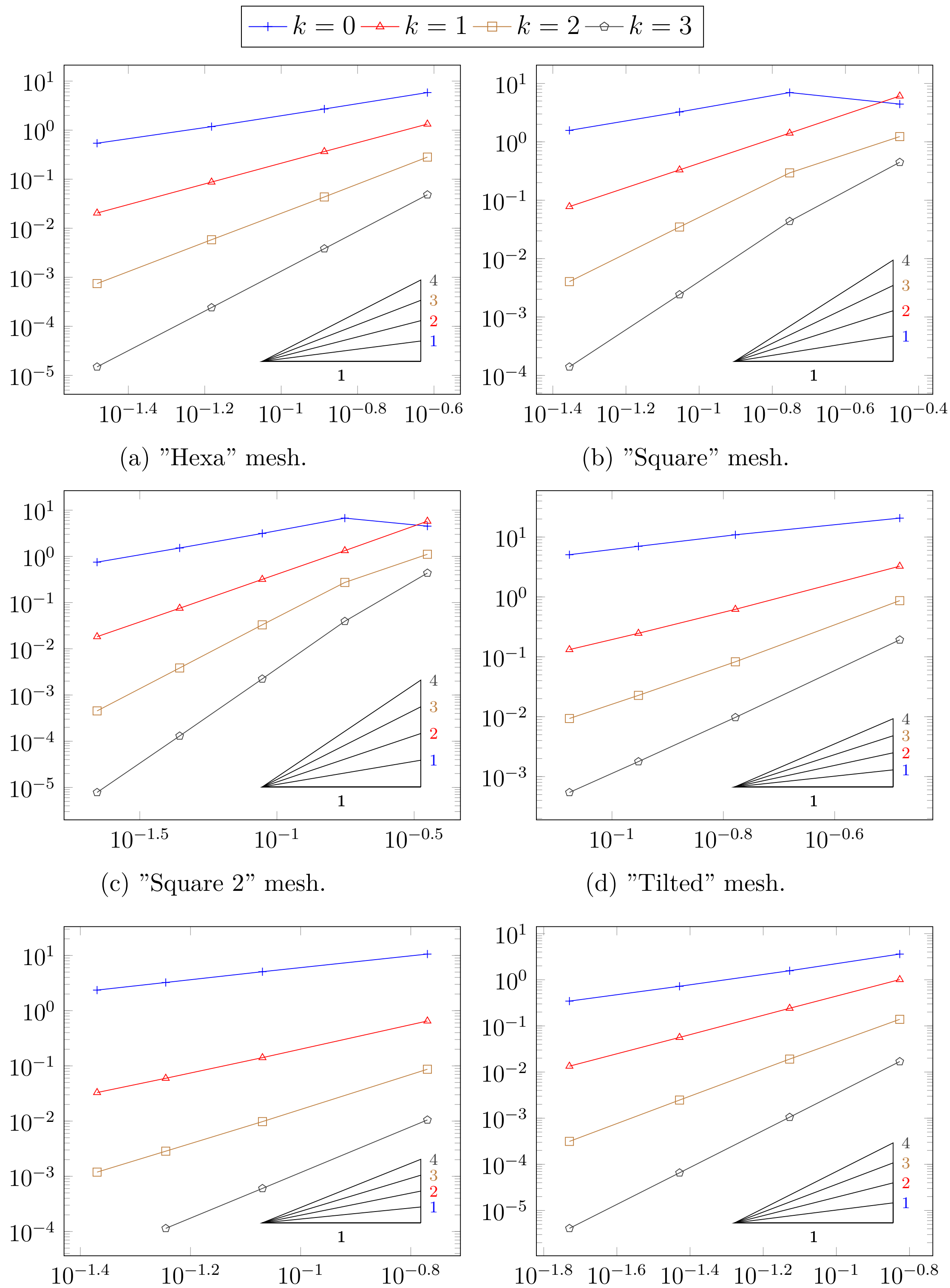

(f) "Hexa anisotropic".

Figure 4.3: Families of meshes used in 2 dimensions.

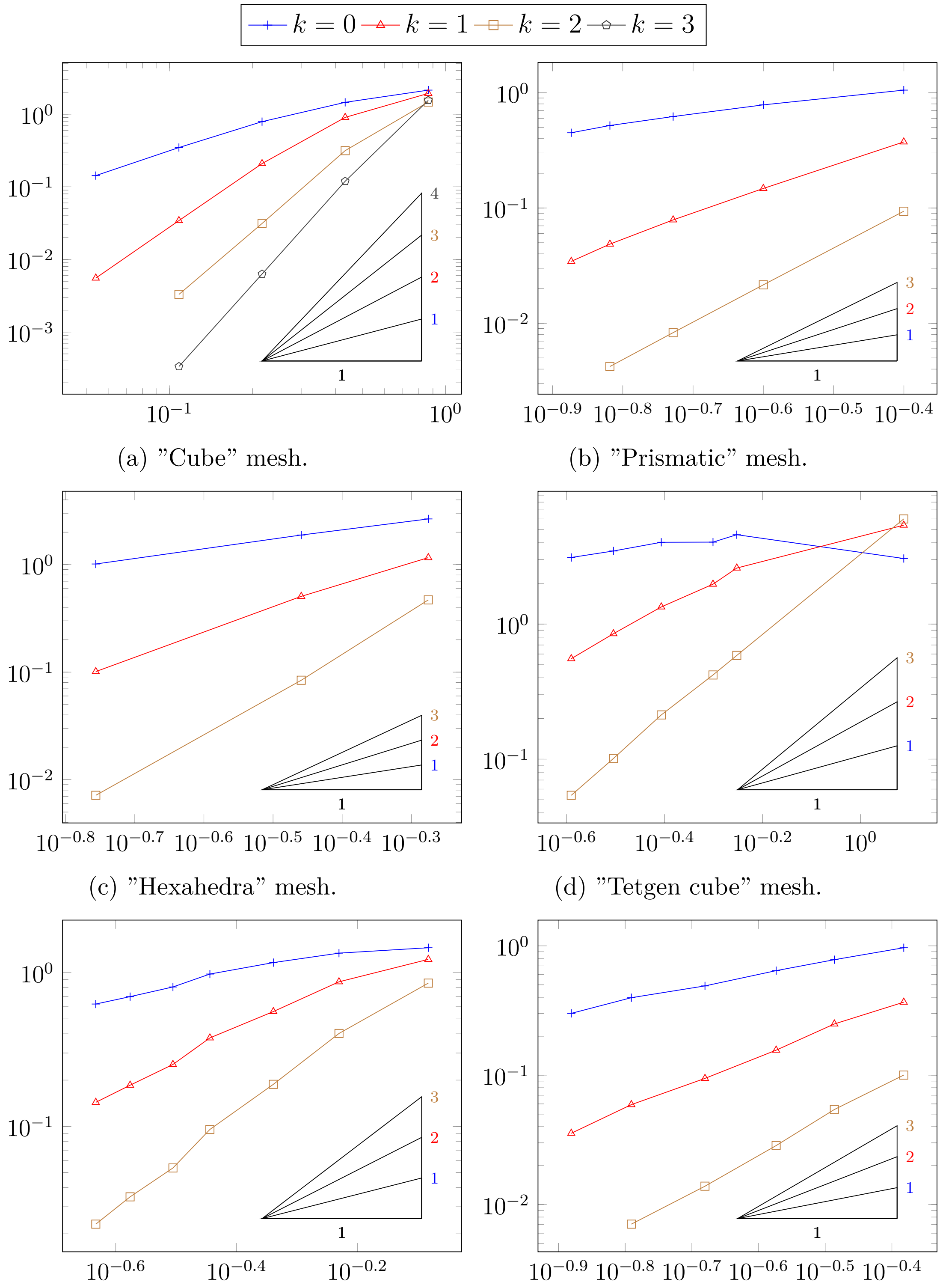


(a) "Hexa" mesh. (b) "Square" mesh.

(c) "Square 2" mesh. (d) "Tilted" mesh.

(e) "Tilted 2" mesh. (f) "Hexa anisotropic" mesh.

Figure 4.4: Absolute error estimate in discrete norm $\|\cdot\|_{\mu,\nabla,1,h} + \|\cdot\|_{L^2,h}$ vs. mesh size h in 2 dimensions.



(a) ”Cube” mesh.

(b) ”Prismatic” mesh.

(c) ”Hexahedra” mesh.

(d) ”Tetgen cube” mesh.

(e) ”Voro” mesh.

(f) ”Voro tets” mesh.

Figure 4.5: Absolute error estimate in discrete norm $\|\cdot\|_{\mu,\nabla,1,h} + \|\cdot\|_{L^2,h}$ vs. mesh size h in 3 dimensions.

Lemma 4.43. *Keeping the notations of* (4.128)*, the subset* $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) = \{W \in (\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3 \,:\, \operatorname{Tr} W = 0\}$ *is characterized by:*

$$\begin{aligned} P_1 &= yzC_1 + y\gamma - z\beta \\ P_2 &= xzC_2 - x\gamma + z\lambda, \\ P_3 &= xyC_3 + x\beta - y\lambda \end{aligned} \qquad \begin{gathered} \lambda \in \mathcal{P}^{k-2}[Y,Z],\ \beta \in \mathcal{P}^{k-2}[X,Z],\ \gamma \in \mathcal{P}^{k-2}[X,Y], \\ C_i \in \mathcal{P}^{k-3}[X,Y,Z],\ C_1 + C_2 + C_3 = 0. \end{gathered}$$

Proof. The proof relies on repeated use of Euclidean division. When we append x (resp. y, z) as subscript we mean that the polynomial does not depend on x (resp. y, z). We write:

$$\begin{aligned} P_2 &= xQ_2 + R_{2,x}, \\ P_3 &= xQ_3 + R_{3,x}. \end{aligned}$$

Then the condition $\operatorname{Tr} W = 0$ becomes $xP_1 + yP_2 + zP_3 = x(P_1 + yQ_2 + zQ_3) + yR_{2,x} + zR_{3,x} = 0$. Looking at the degree in X, we must have $P_1 + yQ_2 + zQ_3 = 0$ and $yR_{2,x} + zR_{3,x} = 0$. Hence there exists $\lambda \in \mathcal{P}^{k-2}[Y,Z]$ such that

$$R_{2,x} = z\lambda,\ R_{3,x} = -y\lambda.$$

On the other hand, writing $P_1 = yA_1 + zA_{1,y} + A_{1,y,z}$, we have $yA_1 + zA_{1,y} + A_{1,y,z} + yQ_2 + zQ_3 = 0$. Looking at the degree in Y and Z we must have $A_{1,y,z} = 0$. Let us write $Q_3 = yB_3 + B_{3,y}$, we have $y(A_1 + Q_2 + zB_3) + z(A_{1,y} + B_{3,y}) = 0$. The degree in Y shows that $A_{1,y} + B_{3,y} = 0$. Thus there exists $\beta \in \mathcal{P}^{k-2}[X,Z]$ such that

$$A_{1,y} = -\beta,\ B_{3,y} = \beta.$$

Let $A_1 = zC_1 + C_{1,z}$ and $Q_2 = zC_2 + C_{2,z}$. We are left with $A_1 + Q_2 + zB_3 = z(C_1 + C_2 + B_3) + C_{1,z} + C_{2,z} = 0$. Once again the degree in Z shows that $C_{1,z} + C_{2,z} = 0$ and $C_1 + C_2 + B_3 = 0$, so there exists $\gamma \in \mathcal{P}^{k-2}[X,Y]$ such that

$$C_{1,z} = \gamma,\ C_{2,z} = -\gamma.$$

We conclude by writing $B_3 := C_3$ and substituting back each term in the development of P_i. □

Remark 4.44. In dimension 2 the expression becomes (see Appendix 4.C):

$$\overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(F) = \begin{pmatrix} -(x-x_F)(y-y_F)Q & -(y-y_F)^2 Q \\ (x-x_F)^2 Q & (x-x_F)(y-y_F)Q \end{pmatrix},\ Q \in \mathcal{P}^{k-1}(F).$$

Lemma 4.45. *For any* $T \in \mathcal{T}_h$*,* $\nabla\cdot$ *is an isomorphism from* $\overline{\boldsymbol{\mathcal{R}}}^{c,k+1}(T)$ *to* $\boldsymbol{\mathcal{G}}^{c,k}(T)$*.*

Proof. By the definition (4.7) the space $\boldsymbol{\mathcal{G}}^{c,k}$ is characterized by:

$$\boldsymbol{\mathcal{G}}^{c,k}(T) = \begin{pmatrix} yQ_1 - zQ_2 \\ zQ_3 - xQ_1 \\ xQ_2 - yQ_3 \end{pmatrix}, \quad Q_1, Q_2, Q_3 \in \mathcal{P}^{k-1}(T). \tag{4.129}$$

Let $W = \begin{pmatrix} xP_1 & yP_1 & zP_1 \\ xP_2 & yP_2 & zP_2 \\ xP_3 & yP_3 & zP_3 \end{pmatrix} \in \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T)$, with P_i given by Lemma 4.43. A direct computation gives

$$\nabla\cdot W = \begin{pmatrix} yzC_1{}' + y\gamma' - z\beta' \\ xzC_2{}' - x\gamma' + z\lambda' \\ xyC_3{}' + x\beta' - y\lambda' \end{pmatrix}, \tag{4.130}$$

with $C_i' = (5 + x\partial_x + y\partial_y + z\partial_z)C_i$, $\gamma' = (4 + x\partial_x + y\partial_y)\gamma$, $\beta' = (4 + x\partial_x + z\partial_z)\beta$, $\lambda' = (4 + y\partial_y + z\partial_z)\lambda$. Because $\xi\partial_\xi$ multiplies the monomial $\xi = x, y, z$ by its power in ξ and the constant preserves the zeroth order term, each of these transformations is a linear automorphism of their respective space (respectively $\mathcal{P}^{k-3}[X, Y, Z]$, $\mathcal{P}^{k-2}[X, Y]$, $\mathcal{P}^{k-2}[X, Z]$ and $\mathcal{P}^{k-2}[Y, Z]$), hence we can drop the $'$ in (4.130) without loss of generality. We must show that there is a one to one correspondence between the Q_i of (4.129) (more precisely their sum, since different Q_i give the same expression) and the C_i, γ, β, λ of (4.130).

Let us write $Q_1 = \gamma + R_1$, $Q_2 = \beta + R_2$ and $Q_3 = \lambda + R_3$ for some R_1, R_2, R_3. The system becomes:

$$\begin{aligned} yzC_1 + y\gamma - z\beta &= yQ_1 - zQ_2 \\ xzC_2 - x\gamma + z\lambda &= zQ_3 - xQ_1 \\ xyC_3 + x\beta - y\lambda &= xQ_2 - yQ_3 \end{aligned} \iff \begin{aligned} yzC_1 &= yR_1 - zR_2 \\ xzC_2 &= zR_3 - xR_1. \\ xyC_3 &= xR_2 - yR_3 \end{aligned}$$

We can check that $xyz(C_1+C_2+C_3) = xyR_1 - xzR_2 + yzR_3 - yxR_1 + zxR_2 - zyR_3 = 0$. Hence given Q_1, Q_2, Q_3, the Euclidean divisions $Q_1 := \gamma + zS_1$, $Q_2 := \beta + yS_2$ and $Q_3 := \lambda + xS_3$ give suitable C_1, C_2, C_3, γ, β, λ for $C_1 = S_1 - S_2$, $C_2 = S_3 - S_1$ and $C_3 = S_2 - S_3$. Likewise, given C_1, C_2, C_3, γ, β, λ, the system

$$\begin{aligned} C_1 &= S_1 - S_2 \\ C_2 &= S_3 - S_1 \\ C_3 &= S_2 - S_3 \end{aligned}$$

is solvable since $C_1 + C_2 + C_3 = 0$ and $Q_1 := \gamma + zS_1$, $Q_2 := \beta + yS_2$, $Q_3 := \lambda + xS_3$ give suitable Q_i. □

Lemma 4.46. *For any $T \in \mathcal{T}_h$, it holds $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) \cap \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T) = \{0\}$.*

Proof. This is a direct consequence of Lemma 4.45 and Remark 4.4. □

Lemma 4.47. *For any $T \in \mathcal{T}_h$, it holds $(\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3 = \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) \oplus \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T)$.*

Proof. Lemma 4.46 already shows that $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) \cap \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T) = \{0\}$. It is enough to compare the dimension of these spaces:

$$\begin{aligned}
\dim \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) &= 3 \dim \mathcal{P}^{k-2}[X,Y] + 2 \dim \mathcal{P}^{k-3}[X,Y,Z] \\
&= 3\frac{k!}{2!(k-2)!} + 2\frac{k!}{3!(k-3)!} = \frac{2k^3+3k^2-5k}{6}, \\
\dim \widehat{\boldsymbol{\mathcal{R}}}^{c,k}(T) = \dim \mathcal{P}^{0,k}(T) &= \frac{(k+3)!}{3!k!} - 1 = \frac{k^3+6k^2+11k}{6}.
\end{aligned}$$

The sum of both is

$$3\frac{(k+2)!}{3!(k-1)!} = 3 \dim \mathcal{P}^{k-1}(T) = \dim (\boldsymbol{\mathcal{R}}^{c,k}(T)^\intercal)^3.$$

□

We now prove the topological decomposition (4.20). We will only prove it for $\overline{\boldsymbol{\mathcal{R}}}^{c,k}(T) \oplus \widehat{\boldsymbol{\mathcal{R}}}^{c,k-1}(T)$, the other cases being similar.

Lemma 4.48. *If $\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T} \in \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T)$ and $\boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T} \in \widehat{\boldsymbol{\mathcal{R}}}^{c,k-1}(T)$ then $\left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}\right\| + \left\|\boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\| \approx \left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T} + \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\|$.*

Proof. Using arguments similar to [67, Lemma 2.2], we can find $\alpha < 1$ depending only on the mesh regularity parameter and the polynomial degree such that $\left|\left|\left|\boldsymbol{\pi}^{c,k}_{\overline{\boldsymbol{\mathcal{R}}},T}\boldsymbol{\pi}^{c,k-1}_{\widehat{\boldsymbol{\mathcal{R}}},T}\right|\right|\right| \leq \alpha$. Then $\forall \boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T} \in \overline{\boldsymbol{\mathcal{R}}}^{c,k}(T)$, $\forall \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T} \in \widehat{\boldsymbol{\mathcal{R}}}^{c,k-1}(T)$,

$$\begin{aligned}
\langle \boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}, \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T} \rangle &= \langle \boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}, \boldsymbol{\pi}^{c,k-1}_{\widehat{\boldsymbol{\mathcal{R}}},T} \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T} \rangle \\
&= \langle \boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}, \boldsymbol{\pi}^{c,k}_{\overline{\boldsymbol{\mathcal{R}}},T} \boldsymbol{\pi}^{c,k-1}_{\widehat{\boldsymbol{\mathcal{R}}},T} \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T} \rangle \\
&\leq \left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}\right\| \left|\left|\left|\boldsymbol{\pi}^{c,k}_{\overline{\boldsymbol{\mathcal{R}}},T}\boldsymbol{\pi}^{c,k-1}_{\widehat{\boldsymbol{\mathcal{R}}},T}\right|\right|\right| \left\|\boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\| \\
&\leq \alpha \left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}\right\| \left\|\boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\| \\
&\leq \frac{1}{2}\alpha^2 \left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}\right\|^2 + \frac{1}{2}\left\|\boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\|^2.
\end{aligned}$$

Therefore

$$\left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T} + \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\|^2 \geq \left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}\right\|^2 + \left\|\boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T}\right\|^2 - 2\langle \boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}, \boldsymbol{v}^c_{\widehat{\boldsymbol{\mathcal{R}}},T} \rangle \geq (1-\alpha^2)\left\|\boldsymbol{v}^c_{\overline{\boldsymbol{\mathcal{R}}},T}\right\|^2.$$

The proof for the other space is similar. □

Lemma 4.49. *For any $T \in \mathcal{T}_h$ and $q \in \mathcal{P}^k(T)$ there is $P \in \mathcal{P}^{k+2|\mathcal{F}_T|}(\overline{T})$ such that $P_{|\partial T} = 0$, $(\mathbf{grad}\, P)_{|\partial T} = \mathbf{0}$, $\pi^k_{\mathcal{P},T} P = q$ and $\|P\| \approx \|q\|$.*

Proof. For any cell $T \in \mathcal{T}_h$ we define the bubble function $\overline{P} := \prod_{F \in \mathcal{F}_T} \left(\frac{\boldsymbol{x}_F - \boldsymbol{x}}{h_T} \cdot \boldsymbol{n}_F \right)^2$, where $\boldsymbol{x}_F$ is a fixed point in F. It follows from the mesh regularity assumption that there exists $h_B \gtrsim h_T$ such that

$$\max_T \overline{P} \lesssim 1, \ \min_{B(\boldsymbol{x}_T, h_B)} \overline{P} \gtrsim 1 \text{ and } \overline{P} \geq 0 \text{ on } T. \tag{4.131}$$

The mapping $(Q, R) \to \int_T \overline{P}\, Q\, R$ is an inner product on $\mathcal{P}^k(T)$, so by the Riesz representation theorem for any q, there is $Q \in \mathcal{P}^k(T)$ such that $\forall R \in \mathcal{P}^k(T)$, $\int_T \overline{P} Q R = \int_T qR$. This shows that $\pi^k_{\mathcal{P},T}(\overline{P} Q) = q$, moreover by construction $(\overline{P} Q)_{|\partial T} = 0$ and $(\mathbf{grad}\, \overline{P} Q)_{|\partial T} = \mathbf{0}$. Using the equivalence of norms [59, Equation 1.43] and (4.131) we have

$$\left\| \overline{P} Q \right\|^2_{L^2(T)} = \int_T \overline{P}^2 Q^2 \approx \int_T \overline{P} Q^2 = \int_T qQ \approx \int_T \overline{P} Q q = \int_T q^2 = \|q\|^2_{L^2(T)},$$

where the hidden constants are independent of Q or T. □

4.B Trace lifting.

In order to prove consistency results we often need functions in Sobolev spaces with suitable properties. We construct them in this section.

Theorem 4.50. *For all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{X}}^k_{\nabla,T}$ there is $\boldsymbol{R}_{\underline{\boldsymbol{v}}_T} \in \boldsymbol{H}^1(T)$ such that*

$$\begin{aligned} \boldsymbol{R}_{\underline{\boldsymbol{v}}_T} &= \widetilde{\gamma^{k+2}_{\nabla,F}}^c \, \underline{\boldsymbol{v}}_F \text{ on } \partial T, \\ \left\| \boldsymbol{R}_{\underline{\boldsymbol{v}}_T} \right\|_T + \left\| \nabla \boldsymbol{R}_{\underline{\boldsymbol{v}}_T} \right\|_T &\lesssim |||\underline{\boldsymbol{v}}_T|||_{\nabla,T} + \left|\left|\left| \underline{\nabla}^{k+1}_T \underline{\boldsymbol{v}}_T \right|\right|\right|_{\boldsymbol{L}^2,T}, \end{aligned} \tag{4.132}$$

where $\widetilde{\gamma^{k+2}_{\nabla,F}}^c \, \underline{\boldsymbol{v}}_F$ is this continuous piecewise polynomial given by Lemma 4.49 such that $\boldsymbol{\pi}^{k+2}_{\boldsymbol{\mathcal{P}},F}(\widetilde{\gamma^{k+2}_{\nabla,F}}^c \, \underline{\boldsymbol{v}}_F) = \gamma^{k+2}_{\nabla,F} \underline{\boldsymbol{v}}_F$, $(\widetilde{\gamma^{k+2}_{\nabla,F}}^c \, \underline{\boldsymbol{v}}_F)_{\partial F} = \mathbf{0}$, and $(\nabla\, \widetilde{\gamma^{k+2}_{\nabla,F}}^c \, \underline{\boldsymbol{v}}_F)_{\partial F} = \mathbf{0}$.

This lift is built upon [23, Theorem 18.40]: Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz continuous of parameters ϵ, L and M (see [23, Definition 13.11]). Then for all $g \in \boldsymbol{B}^{1/2,2}(\partial\Omega)$, there is $c \in \mathbb{R}$ depending only on N and a function $u \in H^1(\Omega)$ such that $\mathrm{Tr}(u) = g$,

$$\|u\|_{L^2(\Omega)} \leq M^{1/2} \epsilon^{1/2} \|g\|_{L^2(\partial\Omega)} \tag{4.133}$$

and

$$\|\mathbf{grad}\, u\|_{\boldsymbol{L}^2(\Omega)} \leq cM(1+L)^{3+N/2}\epsilon^{-1/2}\|g\|_{L^2(\partial\Omega)} + cM(1+L)^{2+(N+1)/2}|g|^{\Diamond}_{\boldsymbol{B}^{1/2,2}(\partial\Omega)}. \tag{4.134}$$

With the Besov seminorm defined by (see [23, Definition 18.36]):

$$|g|^{\Diamond}_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} := \left(\int_{\partial\Omega}\int_{\partial\Omega\cap B(x,\epsilon)} \frac{|g(x)-g(y)|^2}{\|x-y\|^N}dydx\right)^{1/2}. \tag{4.135}$$

Proof of theorem 4.50. We apply the above-mentioned theorem [23, Theorem 18.40] to $\Omega = T$. Here $N = 3$ and the mesh regularity [59, Definition 1.9] allows us to take an open cover of ∂T making it uniformly Lipschitz continuous in the sense of [23, Definition 13.11] with $L = 1$, $M \approx 1$ and $\epsilon \approx h_T$.

Let g be a component of $\widetilde{\gamma^{k+2}_{\nabla,F}}^{c}\,\underline{\boldsymbol{v}}_F$, the matching component of $\boldsymbol{R}_{\underline{\boldsymbol{v}}_T}$ is the u given by (4.133) and (4.134). Without loss of generality we assume that $\int_{\partial T} g = 0$: Else we take instead $\overline{g} = \int_{\partial T} g$ and $u = \overline{g}$ so that $\mathbf{grad}\, u = 0$ and $\|u\|_{L^2(T)} \approx h_T^{\frac{3}{2}}|\overline{g}|$, $\|\overline{g}\|_{L^2(\partial T)} \approx h_T|\overline{g}|$ and u, $\overline{g}$ satisfy (4.132). Hence we reduce to the case $\int_{\partial T} g' = 0$ for $g' = g - \overline{g}$.

Equation (4.133) with the boundedness of the local trace (Lemma 4.22) gives $\left\|\boldsymbol{R}_{\underline{\boldsymbol{v}}_T}\right\|_T \lesssim |||\underline{\boldsymbol{v}}_T|||_{\nabla,T}$ since $\epsilon \approx h_T$. Let $A_F := \frac{1}{|F|}\int_F g_{|F}$ and $A_T := \sum_{F\in\mathcal{F}_T}|F|\,A_F$. By assumption $A_T = 0$, and for all $F \in \mathcal{F}_T$,

$$\begin{aligned}\sum_{F\in\mathcal{F}_T}\left\|g_{|F}\right\|_F &\leq \left\|g_{|F} - A_F\right\|_F + \|A_F - A_T\|_F \\ &\lesssim h_F\left\|\nabla\,\widetilde{\gamma^{k+2}_{\nabla,F}}^{c}\,\underline{\boldsymbol{v}}_F\right\|_F + h_F^{\frac{1}{2}}|||\underline{\boldsymbol{v}}_T|||_{\nabla,T}.\end{aligned}$$

We used the Poincaré-Wirtinger inequality on each face on the first term in the right-hand side and the same proof as Lemma 4.31 (see [67, Equation 5.12]). Hence, Lemma 4.31 gives $\|g\|_{L^2(\partial T)} \lesssim h_T^{\frac{1}{2}}|||\underline{\boldsymbol{v}}_T|||_{\nabla,T}$, and

$$\begin{aligned}\|\mathbf{grad}\, u\|_{L^2(\Omega)} &\lesssim h_T^{-\frac{1}{2}}\|g\|_{L^2(\partial T)} + |g|^{\Diamond}_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} \\ &\lesssim |||\underline{\boldsymbol{v}}_T|||_{\nabla,T}.\end{aligned}$$

We conclude with the estimate on the Besov seminorm Lemma 4.51. □

Lemma 4.51. *Keeping the notations of the proof of Theorem 4.50, it holds*

$$|g|^{\Diamond}_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} \lesssim |||\underline{\boldsymbol{v}}_T|||_{\nabla,T}. \tag{4.136}$$

Proof. We know that $g \in C^1(\mathcal{F}_T)$, hence by the mean value theorem, $\forall y \in B(x,\epsilon)$, $\exists c \in]0,1[$ such that

$$g(\boldsymbol{y}) = g(\boldsymbol{x}) + \mathbf{grad}\, g((1-c)\boldsymbol{x} + c\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x}),$$

thus

$$\frac{|g(\boldsymbol{x}) - g(\boldsymbol{y})|^2}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} = \frac{|\mathbf{grad}\, g((1-c)\boldsymbol{x}+c\boldsymbol{y})|^2 \|\boldsymbol{x}-\boldsymbol{y}\|^2}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} \lesssim \|\mathbf{grad}\, g\|_{L^\infty(F)}^2 |(\boldsymbol{x}-\boldsymbol{y})|^{-1}.$$

Switching to polar coordinates gives

$$\int_{B(x,\epsilon)} \frac{|g(x)-g(y)|^2}{\|x-y\|^3} \lesssim \|\mathbf{grad}\, g\|_{L^\infty(F)}^2 \int_0^\epsilon \frac{1}{r} r \lesssim \|\mathbf{grad}\, g\|_{L^\infty(F)}^2 \epsilon.$$

The discrete inverse inequality (Lemma 4.2) and a Poincaré-Wirtinger inequality show that

$$\begin{aligned}
\left\|\nabla\, \widetilde{\gamma_{\nabla,F}^{k+2}}^{c} \underline{\boldsymbol{v}}_F\right\|_F &\lesssim h_F^{-1} \left\| \widetilde{\gamma_{\nabla,F}^{k+2}}^{c} \underline{\boldsymbol{v}}_F - \frac{1}{|F|}\int_F \widetilde{\gamma_{\nabla,F}^{k+2}}^{c} \underline{\boldsymbol{v}}_F \right\|_F \\
&\approx h_F^{-1} \left\| \gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{v}}_F - \frac{1}{|F|}\int_F \gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{v}}_F \right\|_F \\
&\lesssim \left\|\nabla\, \gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{v}}_F\right\|_F.
\end{aligned}$$

Hence the discrete Lebesgue embedding [59, Lemma 1.25] and Lemma 4.31 give

$$\|\mathbf{grad}\, g\|_{L^\infty(F)}^2 \approx h_F^{-2} \|\mathbf{grad}\, g\|_{L^2(F)}^2 \lesssim h_F^{-3} h_F \left\|\nabla\, \gamma_{\nabla,F}^{k+2}\underline{\boldsymbol{v}}_F\right\|_F^2 \lesssim h_T^{-3} \||\underline{\boldsymbol{v}}_T\||_{\nabla,T}^2.$$

Taking into account $\epsilon \approx h_T$ and $|\partial T| \approx h_T^2$ we find

$$\begin{aligned}
|g|^{\lozenge}_{\boldsymbol{B}^{1/2,2}(\partial\Omega)} &\lesssim \left(\int_{\partial T} \epsilon \|\mathbf{grad}\, g\|_{L^\infty(F)}^2\right)^{\frac{1}{2}} \\
&\lesssim \left(h_T h_T^{-3} \||\underline{\boldsymbol{v}}_T\||_{\nabla,T}^2 \int_{\partial T} 1\right)^{\frac{1}{2}} \\
&\lesssim \||\underline{\boldsymbol{v}}_T\||_{\nabla,T}.
\end{aligned}$$

□

Theorem 4.52. *If $p \in H_0^2(\Omega)$ then there is $\boldsymbol{u} \in \boldsymbol{H}^3(\Omega)$ such that* $\operatorname{div} \boldsymbol{u} = p$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^3} \lesssim \|p\|_{H^2}$, $\|\boldsymbol{u}\|_{\boldsymbol{H}^2} \lesssim \|p\|_{H^1}$ *and* $\|\boldsymbol{u}\|_{\boldsymbol{H}^1} \lesssim \|p\|_{L^2}$.

Proof. Let B be a smooth bounded extension (at least $C^{3,1}$) of Ω. For all function $g \in H^{-1}(B)$, following [30, Theorem III.4.1], there is a unique solution $f \in H_0^1(B)$

to the equation $\Delta f = g$ in B. Moreover this solution satisfies $\|f\|_{H^1} \lesssim \|g\|_{H^{-1}}$ and [30, Theorem III.4.2] shows that if B is $C^{k+1,1}$, $k \geq 0$ and $g \in H^k(B)$ then $\|f\|_{H^{k+2}} \lesssim \|g\|_{H^k}$. Since $p \in H_0^2(\Omega)$ we can extend p by zero and define $\tilde{p} \in H_0^2(B)$ with $\|\tilde{p}\|_{H^k(B)} = \|p\|_{H^k(\Omega)}$, $k \in \{0,1,2\}$ ([8, Theorem 3.33]). Hence, since $\tilde{p} \in H^2(B)$, if we take $f \in H_0^1(B)$ such that $\Delta f = \operatorname{div} \mathbf{grad} f = \tilde{p}$ we get $f \in H^4(B)$ with $\|f\|_{H^2(B)} \lesssim \|p\|_{L^2}$, $\|f\|_{H^3(B)} \lesssim \|p\|_{H^1}$, and $\|f\|_{H^4(B)} \lesssim \|p\|_{H^2}$. Let $\boldsymbol{u} = \mathbf{grad} f_{|\Omega}$ then we have $\operatorname{div} \boldsymbol{u} = p$ in Ω and the expected bounds. □

4.C 2-Dimensional complex.

The Stokes complex also exists in 2 dimensions. Let Ω be a domain of $\mathbb{R}^2$ instead of $\mathbb{R}^3$. The differential complex now reads:

$$\mathbb{R} \xrightarrow{i_\Omega} H^2(\Omega) \xrightarrow{\mathrm{rot}_{\mathrm{F}}} \boldsymbol{H}^1(\Omega) \xrightarrow{\mathrm{div}_{\mathrm{F}}} L^2(\Omega) \xrightarrow{0} \{0\}. \tag{4.137}$$

We can construct a discrete complex similar to the complex in Figure 4.1. However the construction is not merely the restriction of the 3-dimensional complex to 2-dimensional objects. It is in fact much simpler.

Discrete spaces.
The 2-dimensional complex only needs four discrete spaces $\underline{X}^k_{\mathbf{rot},h}$, $\underline{\boldsymbol{X}}^k_{\nabla,h}$, $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h}$ and $\mathcal{P}^k(\mathcal{T}_h)$. They are defined by:

$$\begin{aligned}\underline{X}^k_{\mathbf{rot},h} :=\{\underline{v}_h = ((\boldsymbol{R}_{v,V})_{V\in\mathcal{V}_h}, (v_E, R_{v,E})_{E\in\mathcal{E}_h}, (v_F)_{F\in\mathcal{F}_h}) \,:\, \\ \boldsymbol{R}_{v,V} \in \mathbb{R}^2(V), \forall V \in \mathcal{V}_h, v_E \in \mathcal{P}^{k+1}_c(\mathcal{E}_h), R_{v,E} \in \mathcal{P}^k(E), \forall E \in \mathcal{E}_h, \\ v_F \in \mathcal{P}^{k-1}(F), \forall F \in \mathcal{F}_h\},\end{aligned} \tag{4.138}$$

$$\begin{aligned}\underline{\boldsymbol{X}}^k_{\nabla,h} :=\{\underline{\boldsymbol{w}}_h = ((\boldsymbol{w}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{w}_{\boldsymbol{\mathcal{G}},F}, \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F})_{F\in\mathcal{F}_h}) \,:\, \\ \boldsymbol{w}_E \in \mathcal{P}^{k+2}_c(E;\mathbb{R}^2), \forall E \in \mathcal{E}_h \boldsymbol{w}_{\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^k(F), \boldsymbol{w}^c_{\boldsymbol{\mathcal{G}},F} \in \boldsymbol{\mathcal{G}}^{c,k}(F), \forall F \in \mathcal{F}_h\},\end{aligned} \tag{4.139}$$

$$\begin{aligned}\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h} :=\{\underline{\boldsymbol{W}}_h = ((\boldsymbol{W}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{W}_F)_{F\in\mathcal{F}_h}) \,:\, \\ \boldsymbol{W}_E \in \mathcal{P}^{k+1}(E;\mathbb{R}^2), \forall E \in \mathcal{E}_h, \boldsymbol{W}_F \in \overline{\boldsymbol{\mathcal{RT}}}^{k+1}(F), \forall F \in \mathcal{F}_h\},\end{aligned} \tag{4.140}$$

$$\mathcal{P}^k(\mathcal{T}_h) :=\{\underline{q}_h = ((q_F)_{F\in\mathcal{F}_h}) \,:\, q_F \in \mathcal{P}^k(F), \forall F \in \mathcal{F}_h\}. \tag{4.141}$$

Figure 4.6 is the 2-dimensional equivalent of Figure 4.1.

The interpolator on the space $\underline{X}^k_{\mathbf{rot},h}$ is defined for any $v \in C^1(\overline{\Omega})$ by

$$\underline{I}^k_{\mathbf{rot},h} v = ((v_E, \pi^k_{\mathcal{P},E}(\mathbf{rot_F}\, v \cdot \boldsymbol{t}_E))_{E\in\mathcal{E}_h}, (\mathbf{rot_F}\, v(V))_{V\in\mathcal{V}_h}, (\pi^{k-1}_{\mathcal{P},F}(v))_{F\in\mathcal{F}_h}), \tag{4.142}$$

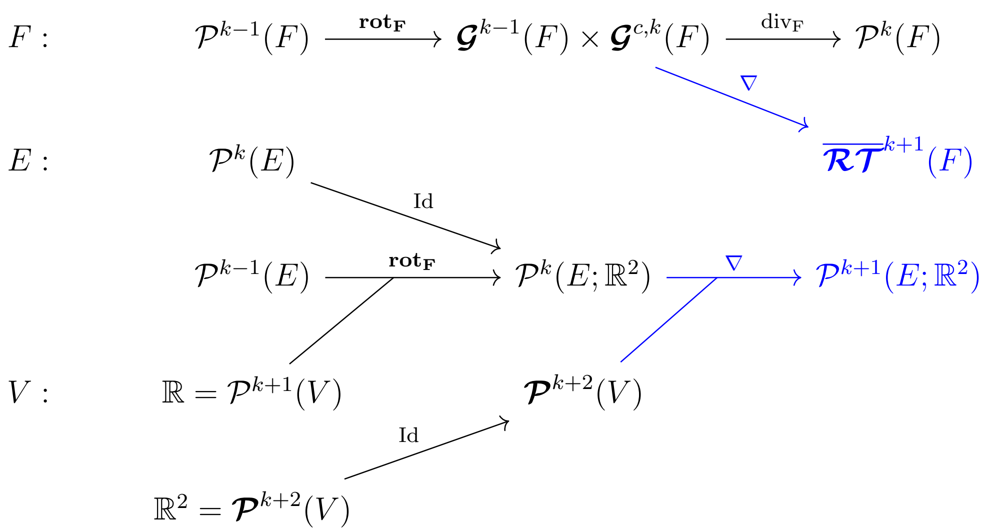


Figure 4.6: Usage of the local degrees of freedom for the discrete differential operators in 2 dimensions.

where for any edge $E \in \mathcal{E}_h$, v_E is such that $\pi^{k-1}_{\mathcal{P},E}(v_E) = \pi^{k-1}_{\mathcal{P},E}(v)$ and for any vertex $V \in \mathcal{V}_E$, $v_E(\boldsymbol{x}_V) = v(\boldsymbol{x}_V)$.

The interpolator on the space $\underline{\boldsymbol{X}}^k_{\nabla,h}$ is defined for any $\boldsymbol{w} \in \boldsymbol{C}^0(\overline{\Omega})$ by

$$\underline{\boldsymbol{I}}^k_{\nabla,h}\boldsymbol{w} = ((\boldsymbol{w}_E)_{E\in\mathcal{E}_h}, (\boldsymbol{\pi}^{k-1}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{w}), \boldsymbol{\pi}^{c,k}_{\boldsymbol{\mathcal{G}},T}(\boldsymbol{w}))_{F\in\mathcal{F}_h}), \tag{4.143}$$

where for any edge $E \in \mathcal{E}_h$, $\boldsymbol{w}_E$ is such that $\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{w}_E) = \boldsymbol{\pi}^k_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{w})$ and for any vertex $V \in \mathcal{V}_E$, $\boldsymbol{w}_E(\boldsymbol{x}_V) = \boldsymbol{w}(\boldsymbol{x}_V)$.

The interpolator on the space $\underline{\boldsymbol{X}}^{k+1}_{\boldsymbol{L}^2,h}$ is defined for any $\boldsymbol{W} \in (\boldsymbol{C}^0(\overline{\Omega})^\intercal)^2$ by

$$\underline{\boldsymbol{I}}^k_{\boldsymbol{L}^2,h}\boldsymbol{W} = ((\boldsymbol{\pi}^{k+1}_{\boldsymbol{\mathcal{P}},E}(\boldsymbol{W}\cdot\boldsymbol{t}_E))_{E\in\mathcal{E}_h}, (\boldsymbol{\pi}^{k+1}_{\overline{\boldsymbol{\mathcal{RT}}},T}(\boldsymbol{W}))_{F\in\mathcal{F}_h}). \tag{4.144}$$

The interpolator on the space $\mathcal{P}^k(\mathcal{F}_h)$ is $\pi^k_{\mathcal{P},\mathcal{F}_h}$, the piecewise L^2-orthogonal projection on spaces $\mathcal{P}^k(F), F \in \mathcal{F}_h$.

Operators and properties.
The discrete operators are defined similarly to the faces operators in Section 4.3.3. Thus all properties of the 3-dimensional complex still applicable hold. Furthermore, the complex property Theorem 4.27 holds (excluding the missing operator $\underline{\boldsymbol{G}}^k_h$ and equation (4.95b)). The consistency results proven in Section 4.5 also hold substituting the faces for the edges and the cells for the faces.

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Résumé. Les complexes différentiels discrets ont récemment attiré l'attention des numériciens en raison des nombreux avantages qu'ils offrent pour la discrétisation des équations aux dérivées partielles. Ils sont particulièrement intéressants pour la création de méthodes préservant la structure telle que par exemple la divergence nulle des solutions discrètes. Dans cette thèse, nous explorons les possibilités d'application de ces complexes aux fluides incompressibles. Nous nous sommes d'abord concentrés sur le complexe de De Rham avec régularité minimale. Bien que très utilisé pour traiter de l'électromagnétisme, il n'a été que très rarement considéré pour les fluides. Nous avons étudié diverses façons d'utiliser ce complexe en mécanique des fluides incompressibles. Nous construisons un schéma pour les équations de Navier-Stokes et l'analysons en démontrant des résultats de convergence, des estimations d'erreur et surtout la préservation de la structure avec une conservation exacte de certaines quantités. La régularité minimale impose cependant des restrictions, et en particulier limite les conditions au bord applicables. Afin de pallier ce problème, nous proposons une discrétisation du complexe de Stokes, autre complexe de régularité plus élevée et mieux adaptée aux équations de Stokes ou Navier-Stokes. Ce dernier complexe repose sur une méthode hybride et entièrement discrète ce qui permet, en plus de profiter des propriétés de complexe, d'utiliser n'importe quel maillage polyédrique (pas forcément conforme ni simplicial).

Mots clés. Calcul extérieur, éléments finis, décomposition de Hodge, complexe de De Rham, complexe de Stokes, Navier-Stokes incompressible, discrétisation compatible, éléments mixtes, Loi de Biot-Savart, complexe de Hodge-Dirac, maillages polyhédriques, ordre arbitraire.

Abstract. Discrete complexes have attracted the attention of numerical scientists because of the many advantages they offer for the discretization of partial differential equations. They are particularly interesting for the creation of structure preserving methods such as methods imposing discrete solutions to be exactly divergence-free. In this thesis, we explore the possible applications of these complexes to incompressible fluids. We first focused on the De Rham complex with minimal regularity. Although it is widely used for electromagnetism, it has been rarely considered for fluids. We wished to study in more depth the possibility of using this complex with fluids. We construct a scheme for the Navier-Stokes equations and analyze it by demonstrating convergence results, error estimates and especially the preservation of the structure with an exact conservation of some quantities. However, the minimal regularity imposes constraints, and in particular limits the applicable boundary conditions. In order to overcome these problems, we design a new discretization of the Stokes complex (which is another complex with higher regularity suitable for Stokes or Navier-Stokes equations) and study its properties. This discrete complex is based on a hybrid and fully discrete method which allows, in addition to take advantage of the complex properties, to use any polyhedral mesh, that is not necessarily conformal nor simplicial.

Keywords. Exterior calculus, finite element, Hodge decomposition, de Rham complex, Stokes complex, incompressible Navier-Stokes, structure preserving discretization, mixed element, Biot-Savart law, Hodge-Dirac complex, polyhedral meshes, arbitrary order.